G-invariant Persistent Homology

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Abstract

Classical persistent homology is not tailored to study the action of transformation groups that are different from the group $Homeo(X)$ of all self-homeomorphisms of a topological space X. In order to obtain better lower bounds for the natural pseudo-distance d_G associated with a group $G \subset Homeo(X)$, we need to adapt persistent homology and consider G-invariant persistent homology. Roughly speaking, the main idea consists in defining persistent homology by means of a set of chains that is invariant under the action of G. In this paper we formalize this idea, and prove the stability of the persistent Betti number functions in G-invariant persistent homology with respect to the natural pseudo-distance d_G. We also show how G-invariant persistent homology could be used in applications concerning shape comparison.

Keywords: Natural pseudo-distance, filtering function, group action, lower bound, stability, shape comparison
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1. Introduction

In many applicative problems we are interested in comparing two $\mathbb{R}^k$-valued functions defined on a topological space, up to a certain group of tranformations. As an example, we can think of the case of taking pictures of two objects A and B from every possible direction and comparing the sets of images we get. In such a case each image can be seen as a point in $\mathbb{R}^k$, and our global measurement as a function $\varphi : S^2 \to \mathbb{R}^k$, taking each direction (represented by a point in $S^2 \subset \mathbb{R}^3$) to the picture we get from that direction. In this case the position of the examined objects cannot be predetermined but we can control the direction of the camera that takes the pictures. As a consequence, two different sets of pictures (described by two different functions $\varphi, \psi : S^2 \to \mathbb{R}^k$) can be considered similar if an orientation-preserving rigid motion g of S^2 exists, such that the picture of A taken from the direction of the unit vector v is similar to the picture of B taken from the direction of the unit vector $g(v)$, for every $v \in S^2$. Formally speaking, the two different sets of pictures can be considered similar if $\inf_{g\in R(S^2)} \max_{v\in S^2} \|\varphi(v) - \psi(g(v))\|_\infty$ is small, where $R(S^2)$ denotes the group of orientation-preserving isometries of S^2.

The previous example illustrates the use of the following definition, where $C^0(X, \mathbb{R}^k)$ represents the set of all continuous functions from X to $\mathbb{R}^k$. These functions are called *k-dimensional filtering functions* on the topological space X.

Definition 1.1. Let X be a triangulable space. Let G be a subgroup of the group $Homeo(X)$ of all homeomorphisms $f : X \to X$. The pseudo-distance $d_G : C^0(X, \mathbb{R}^k) \times C^0(X, \mathbb{R}^k) \to \mathbb{R}$ defined by setting

$$d_G(\varphi, \psi) = \inf_{g\in G} \max_{x\in X} \|\varphi(x) - \psi(g(x))\|_\infty$$

is called the *natural pseudo-distance associated with the group G*.

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The previous definition generalizes the concept of natural pseudo-distance studied in [13, 6, 7, 8, 10] to the case $G \neq Homeo(X)$, and is a particular case of the general setting described in [11]. The case that G is a proper subgroup of $Homeo(X)$ is also examined in [2, 3], and in [12] for the case of the group of diffeomorphisms (in an infinite dimensional setting).

The pseudo-distance d_G is difficult to compute. Fortunately, if $G = Homeo(X)$, Persistent Homology can be used to obtain lower bounds for d_G. For example, if we denote by D_{match} the matching distance between the n-th persistent Betti number functions ρ_n^φ and ρ_n^ψ of the functions φ and ψ, we have that $D_{match}(\rho_n^\varphi, \rho_n^\psi) \leq d_{Homeo(X)}(\varphi, \psi)$ (cf. [1, 5]). For more details about Persistent Homology we refer the reader to [9, 4].

A natural question arises: How could we obtain a lower bound for d_G in the general case $G \neq Homeo(X)$? Does an analogue of the concept of persistent Betti number function exist, suitable for getting a lower bound for d_G? Since $d_{Homeo(X)}(\varphi, \psi) \leq d_G(\varphi, \psi)$, one could think of using the classical lower bounds for the natural pseudo-distance $d_{Homeo(X)}$ in order to get lower bounds for the pseudo-distance d_G. Before proceeding we illustrate an example, showing that in many cases this choice is not useful.

Example 1.2. Let us consider an experimental setting where a robot is in the middle of a room, measuring its distance from the surrounding walls by a sensor, for each direction. This measurement can be formalized by a function $\xi : S^1 \to \mathbb{R}$, where $\xi(v)$ equals minus the distance from the wall in the direction and verse represented by the unit vector v, for each $v \in S^1$. Figure 1 represents two instances φ and ψ of the function ξ for two different shapes of the room. Let $R(S^1)$ denote the group of orientation-preserving rigid motions of $S^1 \subset \mathbb{R}^2$. We observe that a homeomorphism $f : S^1 \to S^1$ exists, such that $\varphi = \psi \circ f$ and $f \notin R(S^1)$. It follows that $d_{Homeo(S^1)}(\varphi, \psi) = 0$, so that classical Persistent Homology cannot give positive lower bounds for $d_{R(S^1)}(\varphi, \psi)$, while we will see that $d_{R(S^1)}(\varphi, \psi) > 0$.

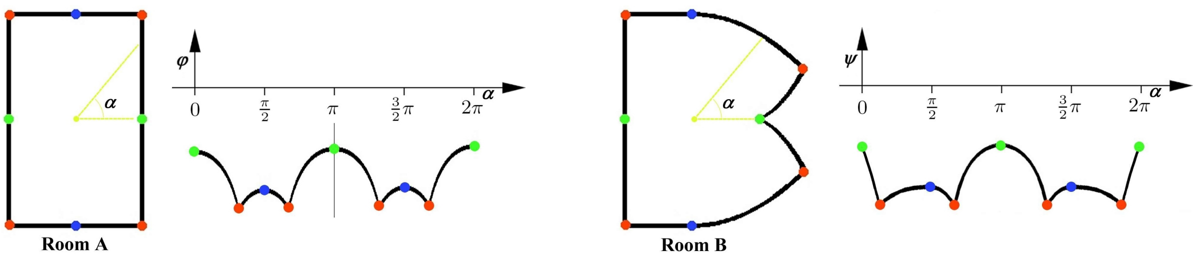

Figure 1: Two rooms and the respective functions φ, ψ, representing minus the distance between the center and the walls. S^1 is identified with the interval $[0, 2\pi]$.

Fortunately, we can adapt Persistent Homology in order to obtain a theory that can give a positive lower bound for d_G, in the previous example (and in many similar cases). We are going to describe this idea in the next section.

2. Adapting Persistent Homology to the group G

Shape comparison is commonly based on comparing properties (usually described by $\mathbb{R}^k$-valued functions) with respect to the action of a transformation group. Let us interpret these concepts in a homological setting. Before proceeding, let us fix a chain complex (C, ∂) over a field $\mathbb{K}$ (so that each group of n-chains C_n is a vector space). We consider the partial order $\preceq$ on $\mathbb{R}^k$ defined by setting $(u_1, \ldots, u_k) \preceq (v_1, \ldots, v_k)$ if and only if $u_j \leq v_j$ for every $j \in \{1, \ldots, k\}$.

Definition 2.1. Assume a function $\bar{\varphi} = (\bar{\varphi}_1, \ldots, \bar{\varphi}_k) : \bigcup_n C_n \to \mathbb{R}^k \cup (-\infty, \ldots, -\infty)$ is given, such that

i) $\bar{\varphi}$ takes the null chain $\mathbf{0} \in C_n$ to the k-tuple $(-\infty, \ldots, -\infty)$, for every $n \in \mathbb{Z}$;

ii) $\bar{\varphi}(\partial c) \preceq \bar{\varphi}(c)$ for every $c \in \bigcup_n C_n$;

iii) $\bar{\varphi}(\lambda c) = \bar{\varphi}(c)$ for every $c \in \bigcup_n C_n$, $\lambda \in \mathbb{K}$, $\lambda \neq 0$;

iv) $\bar{\varphi}_j(c_1 + c_2) \leq \max(\bar{\varphi}_j(c_1), \bar{\varphi}_j(c_2))$ for every $c_1, c_2 \in C_n$ with $n \in \mathbb{Z}$, and every $j \in \{1, \ldots, k\}$.

We shall say that $\bar{\varphi}$ is a *filtering function on the chain complex* (C, ∂).

Definition 2.2. Let us assume that a group G is given, such that G acts linearly on each C_n and its action commutes with ∂, i.e., $\partial \circ g = g \circ \partial$ for every $g \in G$ (in particular, every $g \in G$ is a chain isomorphism from C to C). The chain complex (C, ∂) will be said a *G-chain complex*. We shall call the group $H_n(C) := \ker \partial_n / \text{im } \partial_{n+1}$ the *n-th homology group associated with the G-chain complex* (C, ∂).

Now, let us assume that (C, ∂) is a G-chain complex, endowed with a filtering function $\bar{\varphi}$. For every $u \in \mathbb{R}^k$ we can consider the chain subcomplex $C^{\bar{\varphi} \preceq u}$ of C defined by setting $C_n^{\bar{\varphi} \preceq u} := \{c \in C_n : \bar{\varphi}(c) \preceq u\}$ and restricting ∂ to $C^{\bar{\varphi} \preceq u}$. $C^{\bar{\varphi} \preceq u}$ is a subcomplex of C because of the properties in Definition 2.1 (in particular, $\partial(C_{n+1}^{\bar{\varphi} \preceq u}) \subseteq C_n^{\bar{\varphi} \preceq u}$). We observe that $C^{\bar{\varphi} \preceq u}$ will not be a G-chain complex, since $g(C_n^{\bar{\varphi} \preceq u}) \not\subseteq C_n^{\bar{\varphi} \preceq u}$, in general. For the sake of simplicity, we will use the symbol ∂ in place of $\partial_{|C^{\bar{\varphi} \preceq u}}$.

Definition 2.3. The chain complex $\left(C^{\bar{\varphi} \preceq u}, \partial\right)$ will be called the *chain subcomplex of* (C, ∂) *associated with the value* $u \in \mathbb{R}^k$*, with respect to the filtering function* $\bar{\varphi}$.

We refer to [15] for the definition of chain subcomplex.

Now we can define the concept of the n-th persistent homology group of (C, ∂), with respect to $\bar{\varphi}$.

Definition 2.4. If $u = (u_1, \ldots, u_k), v = (v_1, \ldots, v_k) \in \mathbb{R}^k$ and $u \prec v$ (i.e., $u_j < v_j$ for every index j), we can consider the inclusion i of the chain complex $C^{\bar{\varphi} \preceq u}$ into the chain complex $C^{\bar{\varphi} \preceq v}$. Such an inclusion induces a homomorphism $i^* : H_n\left(C^{\bar{\varphi} \preceq u}\right) \to H_n\left(C^{\bar{\varphi} \preceq v}\right)$. We shall call the group $PH_n^{\bar{\varphi}}(u, v) := i^*\left(H_n\left(C^{\bar{\varphi} \preceq u}\right)\right)$ the *n-th persistent homology group of the G-chain complex C, computed at the point (u, v) with respect to the filtering function $\bar{\varphi}$.* The rank $\rho_n^{\bar{\varphi}}(u, v)$ of this group will be said *the n-th persistent Betti number function (PBNF) of the G-chain complex C, computed at the point (u, v) with respect to the filtering function $\bar{\varphi}$.*

The key property of $PH_n^{\bar{\varphi}}$ is the invariance expressed by the following result.

Theorem 2.5. *If $g \in G$ and $u, v \in \mathbb{R}^k$ with $u \prec v$, the groups $PH_n^{\bar{\varphi} \circ g}(u, v)$ and $PH_n^{\bar{\varphi}}(u, v)$ are isomorphic.*

Proof. We define a map $F : PH_n^{\bar{\varphi} \circ g}(u, v) \to PH_n^{\bar{\varphi}}(u, v)$ in the following way. Let us consider an element $z \in PH_n^{\bar{\varphi} \circ g}(u, v) := i^*\left(H_n\left(C^{\bar{\varphi} \circ g \preceq u}\right)\right)$. By definition, a cycle $c \in C_n^{\bar{\varphi} \circ g \preceq u}$ exists, such that z is the equivalence class $[c]_v$ of c in $H_n\left(C^{\bar{\varphi} \circ g \preceq v}\right)$. We observe that $g(c) \in C_n^{\bar{\varphi} \preceq u}$ and the equivalence class $[g(c)]_v$ of $g(c)$ in $H_n\left(C^{\bar{\varphi} \preceq v}\right)$ belongs to $PH_n^{\bar{\varphi}}(u, v) := i^*\left(H_n\left(C^{\bar{\varphi} \preceq u}\right)\right)$. We set $F(z) = [g(c)]_v$.

If $c' \in C_n^{\bar{\varphi} \circ g \preceq u}$ is another cycle such that $z = [c']_v \in H_n\left(C^{\bar{\varphi} \circ g \preceq v}\right)$, then a chain $\gamma \in C_{n+1}^{\bar{\varphi} \circ g \preceq v}$ exists, such that $c' - c = \partial\gamma$. We observe that $g(\gamma) \in C_{n+1}^{\bar{\varphi} \preceq v}$. The inequality $\bar{\varphi}(\partial(g(\gamma))) \preceq \bar{\varphi}(g(\gamma))$ (see Definition 2.1) implies that $\partial(g(\gamma)) \in C_n^{\bar{\varphi} \preceq v}$. As a consequence, $[g(c')]_v = [g(c + \partial\gamma)]_v = [g(c) + g(\partial\gamma)]_v = [g(c) + \partial(g(\gamma))]_v = [g(c)]_v + [\partial(g(\gamma))]_v = [g(c)]_v$. These equalities follow from the linearity of g and the equality $\partial \circ g = g \circ \partial$ in Definition 2.2. This proves that F is well defined.

Let $z_1 = [c_1]_v, z_2 = [c_2]_v \in PH_n^{\bar{\varphi} \circ g}(u, v)$, with $c_1, c_2 \in C_n^{\bar{\varphi} \circ g \preceq u}$. We observe that $g(c_1), g(c_2) \in C_n^{\bar{\varphi} \preceq u}$. From the linearity of g, it follows that $g(\lambda_1 c_1 + \lambda_2 c_2) = \lambda_1 g(c_1) + \lambda_2 g(c_2) \in C_n^{\bar{\varphi} \preceq u}$, for every $\lambda_1, \lambda_2 \in \mathbb{K}$. Hence, we have that $F(\lambda_1 z_1 + \lambda_2 z_2) = F(\lambda_1 [c_1]_v + \lambda_2 [c_2]_v) = F([\lambda_1 c_1 + \lambda_2 c_2]_v) = [g(\lambda_1 c_1 + \lambda_2 c_2)]_v = \lambda_1 [g(c_1)]_v + \lambda_2 [g(c_2)]_v = \lambda_1 F([c_1]_v) + \lambda_2 F([c_2]_v) = \lambda_1 F(z_1) + \lambda_2 F(z_2)$. Therefore, F is linear.

Furthermore, if $F(z_1) = F(z_2)$ then $[g(c_1)]_v = [g(c_2)]_v$, so that a chain $\hat{\gamma} \in C_{n+1}^{\bar{\varphi} \preceq v}$ exists, such that $g(c_1 - c_2) = g(c_1) - g(c_2) = \partial\hat{\gamma}$. Moreover, $g^{-1}(\hat{\gamma}) \in C_{n+1}^{\bar{\varphi} \circ g \preceq v}$. It follows that $c_1 - c_2 = g^{-1}(\partial\hat{\gamma}) = \partial\left(g^{-1}(\hat{\gamma})\right) \in C_n^{\bar{\varphi} \circ g \preceq v}$, because of Definitions 2.1 and 2.2. As a consequence, $[c_1]_v = [c_2]_v$. This proves that F is injective.

Finally, F is surjective. In order to prove this, we observe that if $w \in PH_n^{\bar{\varphi}}(u, v) := i^*\left(H_n\left(C^{\bar{\varphi} \preceq u}\right)\right)$ with the homomorphism $i^* : H_n\left(C^{\bar{\varphi} \preceq u}\right) \to H_n\left(C^{\bar{\varphi} \preceq v}\right)$ induced by the inclusion $i : C^{\bar{\varphi} \preceq u} \hookrightarrow C^{\bar{\varphi} \preceq v}$, then a chain $\hat{c} \in C_n^{\bar{\varphi} \preceq u}$ exists such that $w = [\hat{c}]_v \in H_n\left(C^{\bar{\varphi} \preceq v}\right)$. We have that $g^{-1}(\hat{c}) \in C_n^{\bar{\varphi} \circ g \preceq u}$ and $F\left([g^{-1}(\hat{c})]_v\right) = [\hat{c}]_v = w$.

Therefore $F : PH_n^{\bar{\varphi} \circ g}(u, v) \to PH_n^{\bar{\varphi}}(u, v)$ is an isomorphism. □

The previous theorem justifies the name *G-invariant Persistent Homology*, showing that the PBNFs of a G-chain complex do not change if we substitute the filtering function $\bar{\varphi}$ with the function $\bar{\varphi} \circ g$, for $g \in G$.

3. Stability of the PBNFs with respect to d_G

Let X and $(S(X), \partial)$ be a triangulable space and its singular chain complex over a field $\mathbb{K}$, respectively.

Assume that a subgroup G of the group $Homeo(X)$ of all homeomorphisms $f : X \to X$ and a continuous function $\varphi = (\varphi_1, \ldots, \varphi_k) : X \to \mathbb{R}^k$ are chosen. For every $u \in \mathbb{R}^k$, let us set $X^{\varphi \preceq u} := \{x \in X : \varphi(x) \preceq u\}$. Let us consider the action of G on $S(X)$ defined by setting $g(\sigma) := g \circ \sigma$ for every $g \in G$ and every singular simplex σ in X, and extending this action linearly on $S(X)$. We recall that, by definition, every singular n-simplex in X is a continuous function from the standard n-simplex into X.

Now, assume that a G-chain subcomplex $(\bar{C}, \partial)$ of the singular chain complex $(S(X), \partial)$ is given. We observe that, for every topological subspace $\bar{X}$ of X, $(\bar{C} \cap S(\bar{X}), \partial)$ is a chain complex over the field $\mathbb{K}$. The symbol $\bar{C} \cap S(\bar{X})$ denotes the chain complex C' where C'_n is the vector space of the singular n-chains in $\bar{X}$ that belong to $\bar{C}_n$.

In order to avoid "wild" chain complexes, we also make this assumption:

($*$) If X' and X'' are two closed subsets of X with $X' \subseteq int(X'')$, then a topological subspace $\hat{X}$ of X exists such that $X' \subseteq \hat{X} \subseteq X''$ and the homology group $H_n(\bar{C} \cap S(\hat{X}))$ is finitely generated.[1]

Let us consider the set $\{\sigma_j^n\}_{j \in J}$ of all (distinct) singular n-simplexes in X. Obviously, if X is not a finite topological space, J will be an infinite (usually uncountable) set. Then we can endow the chain complex $\bar{C}$ with a filtering function $\bar{\varphi}$ in the following way. If c equals the null chain in $\bar{C}_n$, we set $\bar{\varphi}(c) := (-\infty, \ldots, -\infty)$. If c is a non-null singular n-chain, we can write $c = \sum_{r=1}^m a^r \sigma_{j_r}^n \in \bar{C}_n$ with $a^r \in \mathbb{K}$, $a^r \neq 0$ for every index r, and $j_{r'} \neq j_{r''}$ for $r' \neq r''$. In this case we set $\bar{\varphi}(c) = (u_1, \ldots, u_k) \in \mathbb{R}^k$, with each u_i equal to the maximum of φ_i on the union of the images of the singular simplexes $\sigma_{j_1}^n, \ldots, \sigma_{j_m}^n$. In other words, $\bar{\varphi}(c)$ is the smallest vector u such that the corresponding sublevel set $X^{\varphi \preceq u}$ contains the image of each singular simplex $\sigma_{j_r}^n$ involved in the representation of c. We observe that this representation is unique up to permutations of its summands, so that $\bar{\varphi}$ is well defined. Furthermore, the properties in Definition 2.1 are fulfilled. We shall say that the function $\bar{\varphi}$ is *induced by* φ.

An elementary introduction to singular homology can be found in [14].

The next result has a key role in the rest of this paper.

Proposition 3.1. *For every $n \in \mathbb{Z}$ the n-th persistent Betti number function $\rho_n^{\bar{\varphi}}(u, v)$ of the G-chain complex $(\bar{C}, \partial)$, endowed with the filtering function $\bar{\varphi}$, is finite at each point (u, v) in its domain.*

Proof. Since $u \prec v$ and φ is continuous, we have that the set $X^{\varphi \preceq u}$ is closed and contained in the interior of the closed set $X^{\varphi \preceq v}$. Property ($*$) implies that a topological subspace $\hat{X}$ of X exists such that $X^{\varphi \preceq u} \subseteq \hat{X} \subseteq X^{\varphi \preceq v}$ and $H_n(\bar{C} \cap S(\hat{X}))$ is finitely generated. The inclusions $\bar{C} \cap S(X^{\varphi \preceq u}) \xhookrightarrow{i} \bar{C} \cap S(\hat{X}) \xhookrightarrow{j} \bar{C} \cap S(X^{\varphi \preceq v})$ induce the homomorphisms $H_n(\bar{C} \cap S(X^{\varphi \preceq u})) \xrightarrow{i^*} H_n(\bar{C} \cap S(\hat{X})) \xrightarrow{j^*} H_n(\bar{C} \cap S(X^{\varphi \preceq v}))$. Since $\dim \mathrm{im}\,(j^* \circ i^*) \leq \dim \mathrm{im}\, j^* \leq \dim H_n(\bar{C} \cap S(\hat{X})) < +\infty$, we obtain that also $PH_n^{\bar{\varphi}}(u, v) := j^* \circ i^* \left(H_n\left(\bar{C} \cap S(X^{\varphi \preceq u})\right)\right)$ is finitely generated. □

From now on, in order to avoid technicalities that are not relevant in this paper, we shall consider two PBNFs equivalent if they differ in a subset of their domain that has a vanishing measure.

A standard way of comparing two classical persistent Betti number functions is the matching distance D_{match}, a.k.a. bottleneck distance (cf. [9, 5]). It can be applied without any modification to the case of the persistent Betti number functions of the G-chain complex $\bar{C}$. An important consequence of the finiteness

[1] We wish to avoid chain complexes like the one where the 0-chains are all the usual singular 0-chains and the only 1-chain is the trivial one. In this case the homology group $H_0(\bar{C})$ would not be finitely generated, in general. This means that property ($*$) would not hold for $X' = X'' = X$.

of these functions is the following theorem, showing that the matching distance between persistent Betti number functions of the G-chain complex $\bar{C}$ is a lower bound for the natural pseudo-distance d_G. In other words, a small change of the filtering function with respect to d_G produces just a small change of the corresponding persistent Betti number function with respect to D_{match}. This property allows the use of PBNFs in real applications, where the presence of noise is unavoidable.

Theorem 3.2. *For every $n \in \mathbb{Z}$, let us consider the n-th persistent Betti number functions $\rho_n^{\bar{\varphi}}$, $\rho_n^{\bar{\psi}}$ of the G-chain complex $(\bar{C}, \partial)$, endowed with the filtering functions $\bar{\varphi}$ and $\bar{\psi}$ induced by $\varphi : X \to \mathbb{R}^k$ and $\psi : X \to \mathbb{R}^k$, respectively. Then $D_{match}(\rho_n^{\bar{\varphi}}, \rho_n^{\bar{\psi}}) \leq d_G(\varphi, \psi)$.*

Proof. We can proceed by mimicking the proof of stability for ordinary persistent Betti number functions (cf. [5]). This is possible because that proof depends only on properties of PBNFs that are shared by both classical persistent Betti number functions and persistent Betti number functions of a G-chain complex endowed with a filtering function, once we have proven that the PBNFs are finite (Proposition 3.1). It is sufficient to substitute the group $Homeo(X)$ with the group $G \subseteq Homeo(X)$, and the homology groups of each sublevel set $X^{\varphi \preceq u}$ with the homology groups of the G-chain complex $\bar{C} \cap S(X^{\varphi \preceq u})$. □

4. Applications

In this section we illustrate how G-invariant persistent homology can be used to discriminate between the rooms described in Example 1.2, showing that no rotation of S^1 changes the function φ into ψ.

In order to manage this problem we can consider the $R(S^1)$-chain complex $\bar{C}$ whose n-chains are the singular n-chains $c \in S_n(S^1)$ for which the following property holds:

(P) If a singular simplex σ_i^n appears in the representation of c with respect to the basis $\{\sigma_j^n\}$ of $S_n(S^1)$, then the antipodal simplex $s \circ \sigma_i^n$ appears in that representation with the same multiplicity of σ_i^n, where s is the antipodal map $s : S^1 \to S^1$.

In other words, we accept only chains that can be written in the form $\sum_{r=1}^{m} a^r \left(\sigma_{j_r}^n + s \circ \sigma_{j_r}^n\right)$. Every rotation $\rho \in R(S^1)$ commutes with the antipodal map s and is a chain isomorphism from $\bar{C}$ to $\bar{C}$. Moreover, it is easy to verify that the properties in Definition 2.2 are fulfilled, for $G = R(S^1)$ and $C = \bar{C}$. The chains in $\bar{C}$ will be called *symmetric chains*.

We can prove that property $(*)$ holds for the $R(S^1)$-chain complex that we have defined. Let X' and X'' be two closed subsets of S^1 with $X' \subseteq int(X'')$. Let us set $\hat{X}$ equal to the ε-dilation[2] of X' in S^1, choosing $\varepsilon > 0$ so small that the $\hat{X} \subseteq int(X'')$. We observe that the set $\hat{X} \cap s(\hat{X})$ is open and $s\left(\hat{X} \cap s(\hat{X})\right) = \hat{X} \cap s(\hat{X})$. Moreover, $\hat{X} \cap s(\hat{X})$ is the union of a finite family $\mathcal{F} = \{\alpha_i\}$ of pairwise disjoint open arcs, having the property that if $\alpha_i \in \mathcal{F}$ then also $s(\alpha_i) \in \mathcal{F}$ (possibly, $\mathcal{F} = \{S^1\}$). Now, let us consider the topological quotient space Q obtained by taking all unordered pairs of antipodal points in $\hat{X} \cap s(\hat{X})$. We have that Q is homeomorphic to the union of a finite family $\mathcal{F}'$ of pairwise disjoint open arcs of S^1 (possibly, $\mathcal{F}' = \{S^1\}$), and hence the n-th homology group $H_n(Q)$ is finitely generated. A chain isomorphism F from $\bar{C} \cap S\left(\hat{X} \cap s(\hat{X})\right)$ to $S(Q)$ exists, taking each chain $\sigma + s \circ \sigma$ to the chain given by the singular simplex $\{\sigma, s \circ \sigma\}$ in Q. F induces an isomorphism from $H_n\left(\bar{C} \cap S\left(\hat{X} \cap s(\hat{X})\right)\right)$ to $H_n(Q)$. Therefore also $H_n\left(\bar{C} \cap S\left(\hat{X} \cap s(\hat{X})\right)\right)$ is finitely generated. Property $(*)$ follows by observing that $\bar{C} \cap S\left(\hat{X} \cap s(\hat{X})\right) = \bar{C} \cap S(\hat{X})$.

Referring to Example 1.2, we observe that the matching distance between the 0-th persistent Betti number functions of the $R(S^1)$-chain complex $\bar{C}$ with respect to the filtering functions $\bar{\varphi}$ and $\bar{\psi}$ is positive. Hence, Theorem 3.2 gives a non-trivial lower bound for $d_{R(S^1)}(\varphi, \psi)$, while the matching distance between the corresponding classical persistent Betti number functions vanishes. The previous claim becomes clear

[2]The ε-dilation of a subset Y of a metric space M is the set of points of M that have a distance strictly less than ε from Y. On $S^1 \subset \mathbb{R}^2$ we consider the metric induced by the Euclidean metric in $\mathbb{R}^2$.

if we consider the birth of the first homology class in the homology groups $H_0\left(\bar{C}^{\bar{\varphi}\le t}\right)$ and $H_0\left(\bar{C}^{\bar{\psi}\le t}\right)$, respectively, when the parameter t increases. While the group $H_0\left(\bar{C}^{\bar{\varphi}\le t}\right)$ becomes non-trivial when t reaches the value $t_0 = \min\varphi = \min\psi$, the group $H_0\left(\bar{C}^{\bar{\psi}\le t}\right)$ becomes non-trivial when t reaches a value $\bar{t} > \min\varphi = \min\psi$. This is due to the fact that the sublevel set $\{x \in S^1 : \varphi(x) \le t_0\}$ contains two pairs of antipodal points, while the sublevel set $\{x \in S^1 : \psi(x) \le t_0\}$ contains no pair of antipodal points (see Figure 2). By applying Theorem 3.2, it follows that $d_{R(S^1)}(\varphi,\psi) \ge \bar{t} - t_0$.

The interested reader can find the 0-th persistent Betti number functions $\rho_n^{\bar{\varphi}}$ and $\rho_n^{\bar{\psi}}$ of the $R(S^1)$-chain complex $\bar{C}$ in Figure 3.

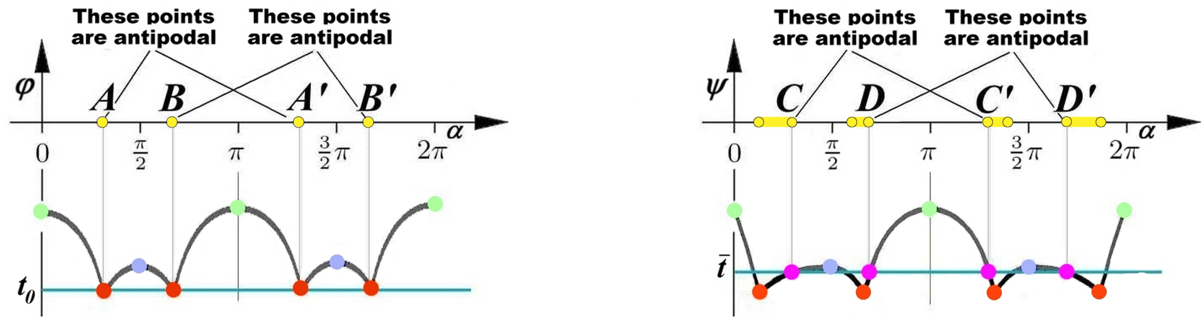

Figure 2: The sublevel sets of the filtering functions φ, ψ cited in Example 1.2, respectively for the levels t_0 and $\bar{t}$.

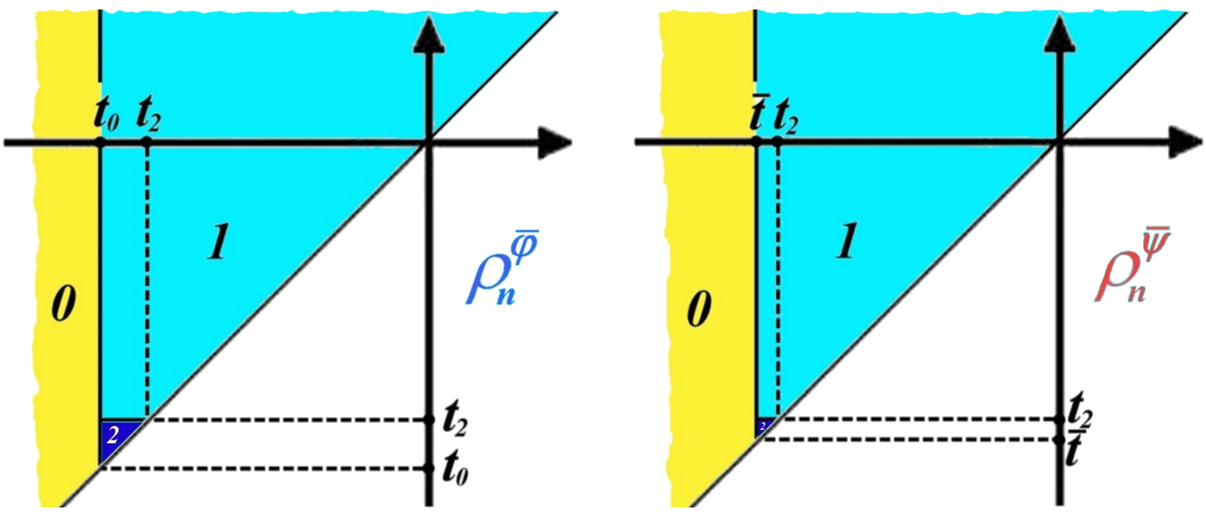

Figure 3: The 0-th persistent Betti number functions $\rho_n^{\bar{\varphi}}$ and $\rho_n^{\bar{\psi}}$ of the $R(S^1)$-chain complex $\bar{C}$, corresponding to the filtering functions φ, ψ cited in Example 1.2. In each part of the domain, the value taken by the PBNF is displayed. Observe that in both figures a small triangle is present, at which the persistent Betti number function takes the value 2.

Remark 4.1. As an alternative approach to the problem of comparing two filtering functions $\varphi, \psi : X \to \mathbb{R}$, the reader could think of using the well known concept of Equivariant Homology (cf. [16]). In other words, in the case that G acts freely on X, one could think of considering the topological quotient space X/G, endowed with the filtering functions $\hat{\varphi}, \hat{\psi}$ that take each orbit ω of the group G to the maximum of φ and ψ on ω, respectively. We observe that this approach would not be of help in the case illustrated in Example 1.2, since the quotient of $S^1/R(S^1)$ is just a singleton. As a consequence, if we considered two filtering functions

$\varphi, \psi : S^1 \to \mathbb{R}$ with $\max \varphi = \max \psi$, the persistent homology of the induced functions $\hat{\varphi}, \hat{\psi} : S^1/R(S^1) \to \mathbb{R}$ would be the same.

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