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Marriage Formation with Assortative Meeting as a Two-Sided Optimal Stopping Problem

Elena M. Parilina
Alessandro Tampieri

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Marriage Formation with Assortative Meeting as a Two-Sided Optimal Stopping Problem*

[Preliminary]

Elena M. Parilina† and Alessandro Tampieri‡

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Abstract

In this paper we study marriage formation through a two-sided secretary problem approach. We consider individuals with nontransferable utility and two different dimensions of heterogeneity, a characteristic evaluated according to the idiosyncratic preferences of potential partners, and an universally-rankable characteristic. There are two possible states of the world, one in which people meet their partner randomly, and one in which the meeting occurs between individuals with similar characteristics. We show that individuals with higher universal characteristic tend to be more picky in their marriage hunting. This does not necessarily mean that they marry later than other individuals, since the higher expected quality of their potential partners in the assortative meeting state can make them marry earlier than individuals with a lower universal characteristic.

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†Faculty of Applied Mathematics and Control Processes, Saint-Petersburg State University, Saint-Petersburg; elena.parilina@gmail.com.

‡Department of Economics, University of Bologna, Strada Maggiore 45, 40125 Bologna, Italy; alessandro.tampieri@unibo.it.

1 Introduction

In economic theory, marriage formation has been studied according to different approaches. A first way is the theory, pioneered by Gale and Shapley (1962) and surveyed and extended by Roth and Sotomayor (1990), who proved the existence of stable matchings. A different perspective is based on the assignment problem (Shapley and Shubik, 1972 and Becker, 1973). In the developments of the assignment problem, the literature borrowed the standard Diamond-Mortensen-Pissarides search framework (see Shimer and Smith, 2000, and Atakan, 2006, *inter alia*).

The underlying assumption of these approaches is that all preferences are known. However, there are many situations in which individuals do not know their own preferences from the beginning. For example, an individual may base his preferences on the potential partner whom he or she meets only. Also, an individual would not be able to meet all the potential partners if they live in another city or country, and even assuming that this may be possible with the new technologies (such as social networks, or dating online services), time is a scarce resource and some individuals will never have the chance of meeting each other, even though they are a perfect match. In these types of situations where only a small portion of preferences will ever be revealed, investigating the best overall matching may not be the most relevant analysis.

In this paper we study marriage formation by assuming that individuals learn their preferences during the search of potential partners. The framework considered is a refined version of the "secretary problem" (Chow *et al.*, 1964), which can be explained as follows. Imagine an administrator who wants to hire the best secretary out of rankable applicants for a position. Applicants are interviewed one-by-one in random order. A decision about each particular applicant is to be taken immediately after the interview. Once rejected, an applicant cannot be recalled. During the interview, the administrator can rank the applicant among all applicants interviewed so far, but is unaware of the quality of yet unseen applicants. The question is about the optimal strategy (stopping rule) to maximize the probability of selecting the best applicant.

Our model differs from the traditional secretary problem for two main elements. First, in order to represent marriage formation, the secretary problem needs to be considered "two-sided", i.e., each partner needs to determine his or her stopping rule (Eriksson *et al.*, 2007).[1] Second, like in the literature of economics of marriage, we consider the possibility

[1] In addition, in the classical secretary problem a player ranks a potential partner by a natural number

that individuals are assortatively matched.

Unlike the stable matching theory, in which each player knows all his/her preferences in advance, in the secretary problem preferences are only revealed slowly as the player meets new potential partners. On the other hand, in the secretary problem we have no guarantees that the marriages will be stable. Compared to the search models, in which potential partners are met with a Poisson arrival (Diamond, 1982, Mortensen, 1982 and Pissarides, 1990), in our approach a meeting with a potential partner takes place in each period.

We consider heterogeneous agents with nontransferable utility,[2] and we try to generalise marriage formation by taking into account two different dimensions of heterogeneity in the characteristics of an individual. Each individual has a characteristic whose evaluation by potential partners depends on the specific idiosyncratic preference of the partner ("specific" characteristic) and another characteristic ("universal" characteristic) that can be ranked in the same way by all individuals, such as income, beauty, social status, and so on.[3]

Another important difference between the present analysis and the standard literature of marriage formation is the way we deal with the relationship between the partners' characteristics. A common element in the marriage formation analysis is the presence of "assortative matching" (Becker, 1973), which alludes to a relationship between the characteristics of partners.[4] In the matching literature, assortative matching is assumed to occur in equilibrium (Becker, 1973 in the seminal paper and Shimer and Smith, 2000, in the search paper, *inter alia*) according to the characteristics of the utility function. In particular, a positive (negative) assortative matching is optimal in equilibrium whenever the utility function is "supermodular" (submodular) in the partners' characteristics, which in words means that the transferable utility function is higher if partners have similar (different) characteristics.[5] In our model, we assume that individuals with similar universal

from $[0, N]$, and two potential partners cannot have the same ranks, while we suppose that the partner's rank is a real number from $[0, 1]$, and a player can meet partners with the same ranks during the game.

[2] The literature of marriage formation consider a family output, which is endogenously shared through a Nash bargaining process. This assumption is called "transferable utility" (see Sattinger, 1995, Lu and McAfee, 1996, Bloch and Ryder, 2000, Shimer and Smith, 2000 and Atakan, 2006). The alternative "non-trasferable utility" indicates that the family output is exogenously shared (see Morgan, 1995, Burdett and Coles, 1997, Chade 2001).

[3] Caldarelli and Capocci (2001) consider a stable matching problem *à la* Gale and Shapley (1962) where partners can be ranked according to a universally classifiable characteristic of an individual.

[4] In particular according to the Becker's model, in equilibrium matching is positively assortative if partners are complements.

[5] From a technical perspective, supermodularity (submodularity) means that, denoting as x and y the

characteristics have a specific probability of meeting due to facts of life (i.e., attending similar social environments, obtaining the same level of education, etc...), even though this does not necessarily lead to marriage formation. In order to distinguish our approach, we will refer to this type of meeting as “assortative meeting”. More specifically, in our framework a meeting can be random (the partner is randomly drawn by the population) or assortative (the potential partner belongs to the same universal rank of the individual) in each period, according to an exogenous and constant probability. From this perspective, the paper offers a comparison of different types of meeting and how these affect the individuals' behaviour.

The results depend on the state of the world in which an individual stands. In assortative meeting, individuals with a high universal characteristic are less demanding if the probability of having assortative meetings in the future is low, and *vice versa*. This result is due to the fact that, given a low probability of being in another assortative meeting state, the quality of the expected future partners is low for individuals with high universal characteristic. Therefore they are less fussy with the choice of a potential partner met today of the same universal rank. In random meeting, an individual with high universal characteristic is harder to please compared to other individuals, and they are more demanding the higher the weight of the universal characteristic. The reason is that an individual with a high universal rank knows that the chance of being in assortative meeting state in the future ensures a high expectation about future meetings, at least for the universal characteristic perspective. This does not necessarily mean that individuals with a high universal characteristic marry later than other individuals. Indeed, individuals with a high universal characteristic expect better-quality partners, which increases the chance of an early marriage.

The remainder of the paper is structured as follows. Section 2 presents the model. Section 3 shows the baseline results. Section 4 illustrates the expected time necessary to marry. Section 5 investigates the case with state-independent strategies. Section 6 concludes. All formal conclusions are derived in the appendix.

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partners' characteristics, and utility being a function of them, $f(x,y)$, then $f(x,y)$ has the following feature: $\frac{\partial^2 f(x,y)}{\partial x \partial y} \propto \frac{\partial^2 f(x,y)}{\partial y \partial x} > (<)\, 0$.

2 The model

We study a large universe of U men and U' women, where $U \gg N$ and $U' \gg N$. Time is discrete, the game starts at period $t = 1$ and lasts for N periods. In each period a meeting takes place. During the meeting, players rank the person of the opposite sex using two characteristics. The first characteristic, denoted by η, reflects the specific, idiosyncratic and universally unrankable traits of an individual. Some individuals like caring and attentive partners, some others prefer independent persons. This is totally subjective and cannot be compared between different individuals. The second characteristic, denoted by I, represents a universally rankable aspect of the individual, such as income, education, social class and so forth.

We assume that the rank of a person is the linear combination of these characteristics:

$$R = (1 - \alpha)\eta + \alpha I, \tag{1}$$

where $\alpha \in (0, 1)$ weights the importance of the universal rank compared to the individual rank. We assume α to be public information and identical for all players. The level of α reflects the role played in the romantic choice by universally estimable characteristics (social class, income, education) compared to personal preferences for specific aspects of a partner. For instance, it can be imagined that in a conservative society individuals put more weight on aspects such as the social status or income when they evaluate a partner.

The meeting can be of two types. We denote the set of types as $S = \{r, \bar{r}\}$, where type $s = r$ is called "random" meeting while $s = \bar{r}$ is called "assortative" meeting. A random meeting occurs when an individual meets the partner by chance. This happens anytime the rankable characteristic of an individual (social status, income, education, etc) plays no role in the occurring meeting. For example, two individuals running into each other at the grocery store, both going to the football stadium or to a public party. Therefore, with random meeting any two people from the universe can meet. Assortative meeting occurs when an individual meets the partner in a contest in which his or her rankable characteristic is relevant in determining the meeting. All the encounters at school, at the university, in a family or a private party are examples of assortative meeting. For the sake of simplicity, we assume that, with assortative meeting, the universal rank of the potential partner will be the same as the individual.

In each period t, the meeting is assortative with exogenous probability $\beta \in (0, 1)$ and random with probability $1-\beta$, β being constant, equal to all the players and known for them. The value of β depends on the customs of the society we have in mind. For instance in a traditional society, it is more likely that individual with common background are matched together (β high).

After each meeting, a man m and a woman w decide whether to propose a marriage to each other. If both propose, the process ends. If at least one of them does not propose, then the game transits to the next period. For simplicity, we assume that being not married is always worse than being married. This assumption implies that, at period N, all the remaining unmatched players are willing to marry.

Since the characteristics of potential partners are not known at the beginning of the game assume that an individual i, $i \in \{m, w\}$ in each period $t = 1, \ldots, N$ meets a partner j, $j \in \{m, w\}$ and $j \neq i$ in state s with the following rank:

$$R_i^{t,s}(I_i) = \begin{cases} (1-\alpha)\,\eta_j^t + \alpha I_j^t, & \text{if } s = r \text{ (with prob. } 1-\beta \text{)} \\ (1-\alpha)\,\eta_j^t + \alpha I_i, & \text{if } s = \bar{r} \text{ (with prob. } \beta \text{)} \end{cases}$$

where

- η_j^t is a random variable with continuous uniform distribution in $[0,1]$ for all $t = 1, \ldots, N$, reflecting the idiosyncratic preference of an individual i for a potential partner j. Let η_j^t be independent variables for $t = 1, \ldots, N$.

- I_j^t is a random variable with continuous uniform distribution in $[0,1]$ for all $t = 1, \ldots, N$, representing the universal rank of a potential partner j. Let I_j^t be independent variables for $t = 1, \ldots, N$.

- $I_i = I \in [0,1]$ is the universal rank of the partner with assortative meeting, which is the same as individual i who evaluates the partner j. The personal rank is known to the individual and does not change throughout the game. This of course is a simplification, as characteristics may change over time, altering I_i. For instance, income generally increases over time, whereas beauty decreases over time.

We assume that men and women rank potential partners symmetrically. This assumption is for the sake of simplicity and does not correspond exactly to what happens in the real world. For instance, in many societies beauty is more evaluated by men, whereas income is more evaluated by women (See Coles and Francesconi, 2011). The assumption that men and women rank potential partners symmetrically implies that the universal rank in assortative meeting state I is equal for man m and woman w.

Consider the following noncooperative game. Each player wants to maximize the expected absolute rank of the chosen mate. The strategy of player i is the rule $a = a(t, s, I_i)$ that says whether the marriage must be proposed to a potential partner with absolute rank $R_j^{t,s}$ and universal rank I_i in period t and in state s for every $t = 1, \ldots, N$. A player's strategy is a set of thresholds such that the player must propose a marriage in period t and in state s if and only if the observed rank is greater than the strategy in t, i.e., $R_i^{t,s} > a(t, s, I_i)$. Therefore a high a implies that a player is more likely to delay marriage, since he or she needs to meet a potential partner with a high rank R in order to agree to marry.

Assume that all players in the game use this type of strategies.

Definition 1 *The N-period process is a N-period meeting game where all players use the same type of threshold strategies, i.e. player i's strategy in period $t = 1, \ldots, N$ and in state s is $a = a(t, s, I_i)$.*

We formulate the problem as a dynamic game and can use the concept of subgame perfect equilibrium (Selten, 1975).

3 Baseline results

3.1 Bellman equation

The following Bellman equation represents the expected partner's rank, for every s using strategy $a = a(t, s, I_i)$, either if a player i marries at t or if he/she waits for the next periods:

$$\begin{aligned} E^{t,s}(a) = &\Pr(\text{marry}|s,a)E[R_i^{t,s}|\text{marry},a] \\ &+\delta(1-\Pr(\text{marry}|s,a))\left(\beta E^{t+1,\bar{r}} + (1-\beta)E^{t+1,r}\right) \end{aligned} \tag{2}$$

with boundary conditions for $t = N$ and states $s = \bar{r}$ and $s = r$:

$$E^{N,\bar{r}} = E[(1-\alpha)\eta_j^N + \alpha I] = \frac{1-\alpha}{2} + \alpha I, \tag{3}$$

$$E^{N,r} = E[(1-\alpha)\eta_j^N + \alpha I_j^N] = \frac{1}{2}. \tag{4}$$

$\Pr(\text{marry}|s, a)$ is the probability of marriage in period t when the state is s and player i's strategy is a, $E[R_i^{t,s}|\text{marry}, a]$ is the expected rank of a potential partner j met in period t in state s if player i marries using strategy a, and $\delta \in (0, 1]$ is the discount factor. Expression $\beta E^{t+1,\bar{r}} + (1-\beta)E^{t+1,r}$ is the expected payoff of an individual i (or absolute rank of j) if they chose to not marry in period t and game transmits to the next period. Notice that player i's strategy $a(t, s, I_i)$ is within interval $[0, 1]$ if $s = r$, but from the interval $[\alpha I_i, \alpha I_i + 1 - \alpha]$ if $s = \bar{r}$. The latter is the interval of possible values of the random variable $R_i^{t,r}$.

In order to solve the Bellman equation, we begin by deriving the conditional probability of marrying according to the occurring state at time t. The result is summarised in the following proposition.

Proposition 1 *The conditional probability to marry in the assortative meeting state for any period $t = 1, \ldots, N-1$ is given by*

$$\Pr\{marry|s = \bar{r}, a\} = \left(1 - \frac{a - \alpha I}{1-\alpha}\right)^2, \tag{5}$$

while the conditional probability to marry in the random meeting state is given by

1. *For $\alpha \geq \frac{1}{2}$:*

$$\Pr\{marry|s = r, a\} = \begin{cases} \left(1 - \dfrac{a^2}{2\alpha(1-\alpha)}\right)^2, & \text{if } a \in [0, 1-\alpha) \\ \left(1 - \dfrac{2a - (1-\alpha)}{2\alpha}\right)^2, & \text{if } a \in [1-\alpha, \alpha) \\ \left(\dfrac{(1-a)^2}{2\alpha(1-\alpha)}\right)^2, & \text{if } a \in [\alpha, 1] \end{cases} \tag{6}$$

2. For $\alpha < \frac{1}{2}$:

$$\Pr\{marry|s = r, a\} = \begin{cases} \left(1 - \dfrac{a^2}{2\alpha(1-\alpha)}\right)^2, & \text{if } a \in [0, \alpha) \\ \left(1 - \dfrac{2a-\alpha}{2(1-\alpha)}\right)^2, & \text{if } a \in [\alpha, 1-\alpha) \\ \left(\dfrac{(1-a)^2}{2\alpha(1-\alpha)}\right)^2, & \text{if } a \in [1-\alpha, 1] \end{cases} \tag{7}$$

The last step in order to derive the Bellman equation is to determine the conditional expectation of the expected rank of a person if player marries. This is summarised in the following proposition.

Proposition 2 *The conditional expectation in the assortative meeting state for any period* $t = 1, \ldots, N-1$ *is given by*

$$E[R_i^{t,\bar{r}}|marry, a] = \frac{\alpha I + 1 - \alpha + a}{2}, \tag{8}$$

whereas the conditional expectation in the random meeting state is given by

1. For $\alpha \geq \frac{1}{2}$*:*

$$E[R_i^{t,r}|marry, a] = \begin{cases} \dfrac{2a^3 - 3\alpha(1-\alpha)}{3a^2 - 6\alpha(1-\alpha)}, & \text{if } a \in [0, 1-\alpha) \\ \dfrac{3a^2 - (1+\alpha+\alpha^2)}{6a - 3(1+\alpha)}, & \text{if } a \in [1-\alpha, \alpha) \\ \dfrac{2a+1}{3}, & \text{if } a \in [\alpha, 1] \end{cases} \tag{9}$$

2. For $\alpha < \frac{1}{2}$*:*

$$E[R_i^{t,r}|marry, a] = \begin{cases} \dfrac{2a^3 - 3\alpha(1-\alpha)}{3a^2 - 6\alpha(1-\alpha)}, & \text{if } a \in [0, \alpha) \\ \dfrac{3a^2 - (3-3\alpha+\alpha^2)}{6a - 3(2-\alpha)}, & \text{if } a \in [\alpha, 1-\alpha) \cdot \\ \dfrac{2a+1}{3}, & \text{if } a \in [1-\alpha, 1] \end{cases} \tag{10}$$

3.2 Players' optimal strategies

We are now in a position to determine a player's optimal strategy through the analysis of the Bellman equation (2). First, we examine separately the two states of the world $\bar{r}$ and r for each period $t = 1, ..., N-1$. Then, we will show the optimal strategy at $t = 1$ for the entire $N-$game process through numerical examples, and examine the expected time of marrying. From now on, we will omit the label i for brevity.

3.2.1 Assortative meeting

First, consider the assortative meeting state $s = \bar{r}$. The Bellman equation (2) is:

$$E^{t,\bar{r}}(a) = \left(1 - \frac{a - \alpha I}{1 - \alpha}\right)^2 \frac{\alpha I + 1 - \alpha + a}{2} + \left(1 - \left(1 - \frac{a - \alpha I}{1 - \alpha}\right)^2\right) \delta E^{t+1}, \qquad (11)$$

where $E^{t+1} = \beta E^{t+1,\bar{r}} + (1-\beta) E^{t+1,r}$ and with boundary conditions (3) and (4). All multipliers in the right hand side part of (11) are nonnegative, so, for each period t from 1 to $N-1$ we investigate $a(t, \bar{r}, I)$ that maximizes $E^{t,\bar{r}}(a)$. The following proposition shows the optimal strategy with assortative meeting.

Proposition 3 *For each $t = 1, \ldots, N-1$, the optimal strategy $a^*(t, \bar{r}, I)$ in the assortative meeting state $s = \bar{r}$ is:*

$$a^*(t, \bar{r}, I) = \begin{cases} \alpha I, & \textit{if } E^{t+1} < \dfrac{4\alpha I + 1 - \alpha}{4\delta}, \\ \dfrac{4\delta E^{t+1} - (\alpha I + 1 - \alpha)}{3}, & \textit{if } \dfrac{4\alpha I + 1 - \alpha}{4\delta} \le E^{t+1} < \dfrac{\alpha I + 1 - \alpha}{\delta}, \\ \alpha I + 1 - \alpha, & \textit{if } E^{t+1} \ge \dfrac{\alpha I + 1 - \alpha}{\delta}. \end{cases} \qquad (12)$$

In Proposition 3, the optimal strategy is higher the higher an individual's universal rank I in cases when $E^{t+1} < \frac{\alpha I}{\delta} + \frac{1-\alpha}{4\delta}$ and $E^{t+1} \ge \frac{\alpha I + 1 - \alpha}{\delta}$. In other words, it is less likely that an individual would accept to marry if he/she is from a high universal rank.

Corollary 1 follows from Proposition 3.

Corollary 1 *In assortative meetings, an individual does not marry anyone before period N if and only if the expected rank E^{t+1} at $t+1$ satisfies:*

$$E^{t+1} \ge \frac{\alpha I + 1 - \alpha}{\delta} \qquad (13)$$

for every $t = 1, \ldots, N-1$.

Condition (13) can be satisfied when the universal rank is very high and the intensity of assortative meeting is also very high. Accordingly, players wait for potential partners with a higher rank in the following meetings. And if inequality (13) is satisfied for every $t = 1, \ldots, N-1$, the player does not marry until the period N participating in assortative meetings.

3.2.2 Random meeting

We turn now to the case with random meeting. In the case in which $\alpha \geq \frac{1}{2}$, the Bellman equation (2) is:

$$E^{t,r}(a) = \begin{cases} \left(1 - \dfrac{a^2}{2\alpha(1-\alpha)}\right)^2 \dfrac{2a^3 - 3\alpha(1-\alpha)}{3a^2 - 6\alpha(1-\alpha)} \\ \qquad + \left(1 - \left(1 - \dfrac{a^2}{2\alpha(1-\alpha)}\right)^2\right) \delta E^{t+1}, & \text{if } a \in [0, 1-\alpha), \\ \left(1 - \dfrac{2a - 1 + \alpha}{2\alpha}\right)^2 \dfrac{3a^2 - (1 + \alpha + \alpha^2)}{6a - 3(1+\alpha)} \\ \qquad + \left(1 - \left(1 - \dfrac{2a - (1-\alpha)}{2\alpha}\right)^2\right) \delta E^{t+1}, & \text{if } a \in [1-\alpha, \alpha), \\ \left(\dfrac{(1-a)^2}{2\alpha(1-\alpha)}\right)^2 \dfrac{2a+1}{3} \\ \qquad + \left(1 - \left(\dfrac{(1-a)^2}{2\alpha(1-\alpha)}\right)^2\right) \delta E^{t+1}, & \text{if } a \in [\alpha, 1] \end{cases} \tag{14}$$

Conversely if $\alpha < \frac{1}{2}$, then the Bellman equation (2) becomes:

$$E^{t,r}(a)=\begin{cases}\left(1-\dfrac{a^2}{2\alpha(1-\alpha)}\right)^2\dfrac{2a^3-3\alpha(1-\alpha)}{3a^2-6\alpha(1-\alpha)} \\ \qquad +\left(1-\left(1-\dfrac{a^2}{2\alpha(1-\alpha)}\right)^2\right)\delta E^{t+1}, & \text{if } a\in[0,\alpha), \\ \left(1-\dfrac{2a-\alpha}{2(1-\alpha)}\right)^2\dfrac{3a^2-(3-3\alpha+\alpha^2)}{6a-3(2-\alpha)} \\ \qquad +\left(1-\left(1-\dfrac{2a-\alpha}{2(1-\alpha)}\right)^2\right)\delta E^{t+1}, & \text{if } a\in[\alpha,1-\alpha), \\ \left(\dfrac{(1-a)^2}{2\alpha(1-\alpha)}\right)^2\dfrac{2a+1}{3} \\ \qquad +\left(1-\left(\dfrac{(1-a)^2}{2\alpha(1-\alpha)}\right)^2\right)\delta E^{t+1}, & \text{if } a\in[1-\alpha,1].\end{cases} \tag{15}$$

with boundary conditions (3) and (4). Proposition 4 describes the optimal strategy with random meeting for each period $t=1,...,N-1$.

Proposition 4 *For each $t=1,\ldots,N-1$, the optimal strategy $a^*(t,s,I)$ in the random meeting state $s=r$ is:*

Case $\alpha\geq\frac{1}{2}$**.**

$$a^*(t,r,I)=\begin{cases}0, & \text{if } E^{t+1}<\dfrac{1}{4\delta} \\ 1-\alpha, & \text{if } \dfrac{1}{4\delta}\leq E^{t+1}<\dfrac{5-19\alpha+11\alpha^2}{6(1-3\alpha)\delta} \\ \dfrac{1+\alpha}{6}+\dfrac{2\delta}{3}E^{t+1}-\gamma_1, & \text{if } \dfrac{5-19\alpha+11\alpha^2}{6(1-3\alpha)\delta}\leq E^{t+1}<\dfrac{5\alpha+1}{6\delta} \\ \dfrac{6\delta E^{t+1}-1}{5}, & \text{if } E^{t+1}\geq\dfrac{5\alpha+1}{6\delta},\end{cases} \tag{16}$$

where $\gamma_1=\frac{\sqrt{16\delta^2(E^{t+1})^2-16\delta E^{t+1}(1-\alpha)+5\alpha^2+6\alpha+5}}{6}$.

Case $\alpha<\frac{1}{2}$**.**

$$
a^*(t,r,I) = \begin{cases} 0, & \textit{if } E^{t+1} < \dfrac{1}{4\delta} \\ \alpha, & \textit{if } \dfrac{1}{4\delta} \leq E^{t+1} < \dfrac{11\alpha^2 - 3\alpha - 3}{6(3\alpha - 2)\delta} \\ \dfrac{2-\alpha}{6} + \dfrac{2}{3}\delta E^{t+1} - \gamma_2, & \textit{if } \dfrac{11\alpha^2 - 3\alpha - 3}{6(3\alpha - 2)\delta} \leq E^{t+1} < \dfrac{6 - 5\alpha}{6\delta} \\ \dfrac{6\delta E^{t+1} - 1}{5}, & \textit{if } E^{t+1} \geq \dfrac{6 - 5\alpha}{6\delta}, \end{cases} \tag{17}
$$

where $\gamma_2 = \frac{\sqrt{16\delta^2(E^{t+1})^2 - 16\delta E^{t+1}(2-\alpha) + 5\alpha^2 - 16\alpha + 16}}{6}$.

3.3 Existence and uniqueness of the equilibrium

Given the assumptions on the Bellman equation considered, the following result holds.

Proposition 5 *In a N-period meeting game there exists a unique subgame perfect equilibrium.*

Proposition 5 can be explained as follows. The N-period meeting game is a finite extensive game. In the model we assume that players participating in the game want to maximize the rank (1) that player will marry. So, the optimal strategy derived by maximizing the expected rank for N-period meeting game is optimal for all players participating the game. The existence of equilibrium in N-period meeting game is straightforward and follows from Selten (1975).

The uniqueness of the subgame perfect equilibrium when all players use optimal strategies $a^* = a^*(t, s, I_i)$, $t = 1, \ldots, N$, $s = r, \bar{r}$ maximizing Bellman function (2) follows from the form of functions used in the right part of (2). In the case of assortative meeting $s = \bar{r}$, then (2) is a continuous function of a with a unique maximum on interval of possible strategy values $[\alpha I, \alpha I + 1 - \alpha]$ for every $t = 1, \ldots, N$. Therefore, each player i has a unique optimal strategy in every period in which assortative meeting takes place. Random meetings can be considered in a similar manner. The function $E^{t,r}(a)$ is continuous in a for both $\alpha \geq \frac{1}{2}$ and $\alpha < \frac{1}{2}$ and has a unique maximum within the interval of possible strategies $[0, 1]$ for every $t = 1, \ldots, N$. Hence, a player has a unique optimal strategy in every period in which random meeting occurs.

3.4 Analysis of equilibrium

In this section we consider properties of optimal strategies in both random and assortative meetings.

Proposition 6 *The equilibrium payoff in the assortative (random) meeting state is an increasing (non decreasing) function of a player's individual rank I.*

Proposition 7 *If the following condition holds:*

$$\beta > \frac{1}{4\delta^t}, \tag{18}$$

for any $t = 1, \ldots, N$, then in the assortative meeting state, $s = \bar{r}$, the optimal strategy $a^*(t, \bar{r}, I)$ is a non-decreasing function of a player's universal rank I for any t.

Condition (18) is sufficient but not necessary. Indeed the necessary condition for non-decreasing function $a^*(t, \bar{r}, I)$ of I is very difficult to be obtained in explicitly, this due to the recurrent form of optimal strategies. For example, for $t = N - 1$ the condition $\beta > \frac{1}{4\delta}$ is also necessary. For $t = N - 2$ the necessary condition is:

$$\beta > \frac{1 - \alpha + \alpha I - \frac{1-\alpha}{4}\sqrt{\frac{3(1-\delta)}{\delta}}}{\delta\alpha},$$

and so on.

Proposition 7 shows that, in the assortative meeting state, the optimal strategy changes with a player's universal rank according to the intensity of assortative meeting. If β is high, players with high universal rank are more "demanding", because the future chance of being in the assortative meeting state (and thus to meet high ranked partners) will be higher. Therefore they can wait for a better idiosyncratic match. Conversely, if β is low, then players are more picky if they have a low I, since a low β implies a relatively higher future expectations for low-I types. Indeed low universal rank players obtain a higher payoff from a random meeting.

The next propostion shows how the optimal strategy changes according to the universal rank of a player in the random meeting state. In the random meeting state, the high-I types generally are more patient, as their future potential partners generally have a higher expected rank, due to the chance of being in the assortative meeting state.

Proposition 8 *For the random meeting state $s = r$, the optimal strategy $a^*(t, r, I)$ is a non-decreasing function of a player's individual rank I.*

Finally, we examine the effects of a variation of the intensity of assortative meeting β. Notice that E^{t+1} is a function of β, in particular

$$\frac{\partial}{\partial \beta} E^{t+1}(\beta) \lesseqqgtr 0 \text{ as } E^{t+1,\bar{r}} \lesseqqgtr E^{t+1,r} \tag{19}$$

Therefore the effect of β on the optimal strategy in both assortative and random meeting state depends on which future conditional expectation is higher. Since the value of $E^{t+1,\bar{r}}$ strictly depends on the individual's universal rank, then in turn it is more likely that $E^{t+1,\bar{r}} > E^{t+1,r}$ (and in turn $\frac{\partial}{\partial \beta} E^{t+1} > 0$) for higher levels of I.

4 Expected number of periods needed to marry

In this section we examine how long a player remains unmarried. We denote T as a discrete random variable representing the number of the periods in which a player expects to marry, where $T = 1, 2, ..., N$. For calculating the mathematical expectation of the number of periods needed to marry we need to find the probabilities that a player marries in each particular period t. Denote this probability as P_t, $\forall\ t = 1, \ldots, N$. For period 1, this probability can be defined by the following expression:

$$P_1 = (1-\beta)\Pr\{\text{marry}|s = r, a(1, r, I)\} + \beta \Pr\{\text{marry}|s = \bar{r}, a(1, \bar{r}, I)\} \equiv M_1. \tag{20}$$

For period 2, the probability to marriage is as follows:

$$\begin{aligned} P_2 = &(1 - M_1)\left((1-\beta)\Pr\{\text{marry}|s = r, a(2, r, I)\}\right. \\ &\left.+ \beta \Pr\{\text{marry}|s = \bar{r}, a(2, \bar{r}, I)\}\right) = (1 - M_1)M_2. \end{aligned} \tag{21}$$

For period k, the probability can be obtained by the expression:

$$\begin{aligned} P_k = &(1 - M_1)\ldots(1 - M_{k-1})\left((1-\beta)\Pr\{\text{marry}|s = r, a(k, r, I)\}\right. \\ &\left.+ \beta \Pr\{\text{marry}|s = \bar{r}, a(k, \bar{r}, I)\}\right) = (1 - M_1)\ldots(1 - M_{k-1})M_k. \end{aligned} \tag{22}$$

If a player does not marry in the first $N-1$ periods of the game and participates in the last Nth period he marries in this period with probability 1 because of the assumption that player always prefers to get married than to be single, i.e.

$$P_N = (1 - M_1)\dots(1 - M_{N-1}).$$

We can determine the expectation of T as follows.

Proposition 9 *The expected number of periods it takes an individual to become married is given by:*

$$ET = P_1 + 2P_2 + \dots + NP_N = \sum_{i=1}^{N} i \left\{ \prod_{k=1}^{i-1} (1 - M_k) \right\} M_i.$$

The expected number of periods needed to marry is a function of the player's strategy a and all parameters of the game α, β, I.

Using a numerical simulation we examine the number of periods needed to marry. First we consider it for different universal ranks. We appoint the following parameters values: $\beta = 0.7$, $N = 100$, $\delta = 1$, $I = 0.01$, 0.33, 0.66, and 0.99. Consider first $\alpha = 0.25$ (Table 1). In this case, the higher the rank of a player, the less the expected time of marrying. This result can be explained as follows. A player with high universal rank tends to have higher expectations about future matches. This is due to the chance of being in the assortative meeting state. Indeed, this increases the likelihood that the player meets a potential partner with the same universal rank. Therefore a player with high rank is generally more "demanding" about a partner type. Nonetheless, the chance of being in the assortative meeting state for a high-universal rank individual also has the effect of increasing the quality of each meeting. As a consequence, a player with high universal rank may in fact marry sooner than other individuals as the second effect can offset the first one.

For $\alpha = 0.80$, the relationship between universal rank and time to marry is non-monotone: the time to marry is low for individuals with low universal rank, it increases for medium levels of universal rank and it decreases again for high universal rank. Two factors contribute to obtain this. First, the higher importance of α makes individuals with a high universal characteristic to be more picky in their partner choice, thus delaying the marriage. The first effect prevails on the second effect when the universal characteristic is not so high, but for very high universal characteristic the second effect more than offsets the first effect, so that the time expected of marrying is lower. Thus individuals with a very high universal characteristic tend to marry sooner than other individuals. Alternatively, an individual with medium-high rank tend to marry later because they are choosy and the quality of individuals they meet is more likely to be lower.

I	$\alpha = 0.25$	$\alpha = 0.80$
0.01	46.38	42.36
0.33	44.60	42.82
0.66	42.48	50.53
0.99	42.27	41.55

Table 1: Expected number of periods needed to marry for different I

Finally, we consider the change in the expected number of periods before marrying for different β. We appoint the following parameters values: $I = 0.9$, $N = 100$, $\delta = 1$, $\beta = 0.01$, 0.33, 0.66, and 0.99. As in the previous example, we assume either $\alpha = 0.25$ or $\alpha = 0.80$ As shown by Table 2, the effect of a variation of β is qualitatively similar to the effect of a variation of I. This seems intuitive, considering that an increase of β relatively increases the importance of I in an individual's payoff.

β	$\alpha = 0.25$	$\alpha = 0.80$
0.01	44.70	43.24
0.33	43.66	41.90
0.66	42.42	42.01
0.99	41.81	41.80

Table 2: Expected number of periods needed to marry for different β

5 State-independent strategies

In this section we modify the N-period meeting game as follows. Suppose that, for every $t = 1, \ldots, N$, a player i uses the same strategy $a(t)$ in assortative and random meetings, so that $a(t) = a(t, \bar{r}, I) = a(t, r, I)$. This situation reflects the situations in which an individual does not know exactly which type of meeting (state) that takes place in every period.

In this modified meeting game we consider the payoff of the player in N-period game, as the linear combination of the player's expected payoffs in the games beginning with particular meetings (assortative and random):

$$E^1(a(1)) = \beta E^{1,\bar{r}}(a(1)) + (1-\beta)E^{1,r}(a(1)).$$

The Bellman equation for the payoff $E^t(a(t))$ takes the form of:

$$\begin{aligned} E^t(a(t)) =& \beta \Pr(\text{marry}|\bar{r}, a(t))E[R^{t,\bar{r}}|\text{marry}, a(t)] \\ &+(1-\beta)\Pr(\text{marry}|r, a(t))E[R^{t,r}|\text{marry}, a(t)] \\ &+\delta\left\{\beta(1-\Pr(\text{marry}|\bar{r}, a(t))) + (1-\beta)(1-\Pr(\text{marry}|r, a(t)))\right\} E^{t+1}(a(t+1)), \end{aligned} \tag{23}$$

with boundary condition:

$$E^N = \beta\left(\frac{1-\alpha}{2} + \alpha I\right) + \frac{1-\beta}{2}. \tag{24}$$

With state-independent strategies, a player uses the same strategies for random and assortative meetings in the same period. Then, the set of possible strategies are in the set $[0, 1]$ for all states. The probability to marry is given by:

$$\Pr\{\text{marry}|s = \bar{r}, a\} = \begin{cases} 1, & \text{if } a \in [0, \alpha I), \\ \left(1 - \dfrac{a-\alpha I}{1-\alpha}\right)^2, & \text{if } a \in [\alpha I, \alpha I + 1 - \alpha), \\ 0, & \text{if } a \in [\alpha I + 1 - \alpha, 1]. \end{cases} \tag{25}$$

Moreover in the assortative meeting state, the conditional expectation of the absolute rank of the chosen j under the condition that the marriage takes place in period t is:

$$E[R_i^{t,\bar{r}}|\text{marry}, a] = \begin{cases} \alpha I + \dfrac{1-\alpha}{2}, & \text{if } a \in [0, \alpha I), \\ \dfrac{\alpha I + 1 - \alpha + a}{2}, & \text{if } a \in [\alpha I, \alpha I + 1 - \alpha), \\ 0, & \text{if } a \in [\alpha I + 1 - \alpha, 1]. \end{cases} \tag{26}$$

With state-independent strategies, the player's optimal strategy is implicitly defined. Notice that the player's payoff in the N-period meeting game with state-independent strategies, i.e. the expected rank of the potential partner, is not larger than the payoff in the game with state-dependent strategies.

6 Concluding remarks

We have studied marriage formation through a two-sided secretary problem approach, where individuals have two different dimensions of heterogeneity, and two possible types of meetings, a random and an assortative one, may occur over time. We show that individuals with higher universal characteristic tend to be more picky in their marriage hunting. This does not necessarily mean that they marry later than other individuals, since the higher expected quality of their potential partners in the assortative meeting state can make them marry earlier than individuals with lower universal characteristic.

The analysis carried out did not consider divorce explicitly, but this indeed can be easily implemented. Once assumed that divorce occurs with exogenous probability, then there is no reason to expect that this probability may change according to whether two individuals decide to marry or not in a certain period, apart from the fact that of course the probability of divorcing increases with the length of a relationship.

A further development may take into account different universal characteristics for men and women. According to the customs considered, these may change according to gender. For example in western societies, men appoint a higher value to beauty compared to women, whereas women appoint a higher value to financial security (See Coles and Francesconi, 2011). Finally, it would be interesting to consider the presence of gays in the two populations, and see how this changes the results. These developments of the current model are left for future works.

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Appendix

Proof of Proposition 1

In order to determine the conditional probability to marry, it is necessary first to obtain the probability distribution of the potential partner's rank.

Probability density and cumulative distribution functions

Assortative meeting: $s = \bar{r}$ We find the probability density distribution function $f_{R_i^{t,\bar{r}}}(x)$ of $R_i^{t,\bar{r}} = (1-\alpha)\,\eta_j^t + \alpha I$ by using the consolidation formula of independent random variables:

$$f_{R_i^{t,\bar{r}}}(x) = \frac{1}{1-\alpha} f_{\eta_j^t}\left(\frac{x-\alpha I}{1-\alpha}\right) = \begin{cases} \dfrac{1}{1-\alpha}, & \text{if } x \in [\alpha I, \alpha I + 1 - \alpha] \\ 0, & \text{if } x \notin [\alpha I, \alpha I + 1 - \alpha], \end{cases} \tag{27}$$

where $f_{\eta_j^t}(x)$ is a probability density function of the variable η_j^t. Thus the cumulative distribution function $F_{R_i^{t,\bar{r}}}(x) = \Pr\{R_i^{t,\bar{r}} \leq x\} = \int_{-\infty}^{x} f_{R_i^{t,\bar{r}}}(u)du$ of the random variable $R_i^{t,\bar{r}}$ is as follows:

$$F_{R_i^{t,\bar{r}}}(x) = \begin{cases} 0, & \text{if } x \in (-\infty, \alpha I) \\ \dfrac{x-\alpha I}{1-\alpha}, & \text{if } x \in [\alpha I, \alpha I + 1 - \alpha) \\ 1, & \text{if } x \in [\alpha I + 1 - \alpha, \infty) \end{cases} \tag{28}$$

Therefore, the linear transformation of η_j^t keeps the same distribution type but changes the interval of possible values, i.e. the distribution of rank $R_i^{t,\bar{r}}$ is continuous uniform in interval $[\alpha I, \alpha I + 1 - \alpha]$.

Random meeting: $s = r$ A player i ranks a potential partner j as follows: $R_i^{t,r} = (1-\alpha)\eta_j^t + \alpha I_j^t$. Here the random variables η_j^t and I_j^t, $t = 1, \ldots, N$ are independent and have the same uniform continuous distribution on the interval $[0, 1]$. The expression for the probability density distribution function $f_{R_i^{t,r}}(x)$ of a random variable $R_i^{t,r}$ can be found using the formula of consolidation of two continuous independent variables:

- Case $\alpha \geq \frac{1}{2}$:

$$f_{R_i^{t,r}}(x) = \int_{-\infty}^{\infty} f_{(1-\alpha)\eta_t}(u) f_{\alpha I_j^t}(x-u)du = \tag{29}$$

$$= \begin{cases} \dfrac{x}{\alpha(1-\alpha)}, & \text{if } x \in [0, 1-\alpha) \\ \dfrac{1}{\alpha}, & \text{if } x \in [1-\alpha, \alpha) \\ \dfrac{1-x}{\alpha(1-\alpha)}, & \text{if } x \in [\alpha, 1] \\ 0, & \text{if } x \notin [0, 1] \end{cases}$$

- Case $\alpha < \frac{1}{2}$:

$$f_{R_i^{t,r}}(x) = \begin{cases} \dfrac{x}{\alpha(1-\alpha)}, & \text{if } x \in [0, \alpha) \\ \dfrac{1}{1-\alpha}, & \text{if } x \in [\alpha, 1-\alpha) \\ \dfrac{1-x}{\alpha(1-\alpha)}, & \text{if } x \in [1-\alpha, 1] \\ 0, & \text{if } x \notin [0, 1] \end{cases} \tag{30}$$

For $s = r$, we find the expression of cumulative distribution function $F_{R_i^{t,r}}(x)$ of random variable $R_i^{t,r}$ according to the value of parameter α:

- Case $\alpha \geq \frac{1}{2}$:

$$F_{R_i^{t,r}}(x) = \begin{cases} 0, & \text{if } x \in (-\infty, 0) \\ \dfrac{x^2}{2\alpha(1-\alpha)}, & \text{if } x \in [0, 1-\alpha) \\ \dfrac{2x-(1-\alpha)}{2\alpha}, & \text{if } x \in [1-\alpha, \alpha) \\ 1 - \dfrac{(1-x)^2}{2\alpha(1-\alpha)}, & \text{if } x \in [\alpha, 1) \\ 1, & \text{if } x \in [1, \infty) \end{cases} \tag{31}$$

- Case $\alpha < \frac{1}{2}$:

$$F_{R_i^{t,r}}(x) = \begin{cases} 0, & \text{if } x \in (-\infty, 0) \\ \dfrac{x^2}{2\alpha(1-\alpha)}, & \text{if } x \in [0, \alpha) \\ \dfrac{2x-\alpha}{2(1-\alpha)}, & \text{if } x \in [\alpha, 1-\alpha) \\ 1 - \dfrac{(1-x)^2}{2\alpha(1-\alpha)}, & \text{if } x \in [1-\alpha, 1) \\ 1, & \text{if } x \in [1, \infty) \end{cases} \tag{32}$$

Notice that in the case of random meeting $s = r$ the distribution of rank $R_i^{t,r}$ is not uniform.

Conditional probability

Given the probability density and the cumulative distribution functions, we are now able to determine the conditional probabilities to marry. We consider the two cases according to $\alpha \geq \frac{1}{2}$, $\alpha < \frac{1}{2}$, and we find the expressions of probability to marry $\Pr\{\text{marry}|s, a\}$ under the condition that the state is s and a player i uses strategy a. This is the probability that both players i and j who met in period t choose each other under the condition that their choices are independent and they both use the same type of strategies.

If the meeting is assortative ($s = \bar{r}$), the conditional probability to marry is as follows:

$$\Pr\{\text{marry}|s = \bar{r}, a\} = \Pr\left\{\left(R_i^{t,\bar{r}} > a(t, \bar{r}, I)\right) \cap \left(R_j^{t,\bar{r}} > a(t, \bar{r}, I)\right)\right\}, \tag{33}$$

where the events $R_i^{t,\bar{r}} > a(t, \bar{r}, I)$ and $R_j^{t,\bar{r}} > a(t, \bar{r}, I)$ are independent, so that:

$$\Pr\{\text{marry}|s = \bar{r}, a\} = \Pr^2\left\{R_i^{t,r} > a(t, r, I_i)\right\} = \left(1 - \frac{a - \alpha I}{1 - \alpha}\right)^2, \tag{34}$$

where $a = a(t, \bar{r}, I) \in [\alpha I, \alpha I + 1 - \alpha]$. In the case of random meeting ($s = r$), this probability is given by:

$$\Pr\{\text{marry}|s = r, a\} = \Pr^2\left\{R_i^{t,r} > a(t, r, I_i)\right\} = \left(1 - F_{R_i^{t,r}}(a(t, r, I_i))\right)^2 \tag{35}$$

- For $\alpha \geq \frac{1}{2}$:

$$\Pr\{\text{marry}|s=r,a\} = \begin{cases} \left(1 - \dfrac{a^2}{2\alpha(1-\alpha)}\right)^2, & \text{if } a \in [0, 1-\alpha) \\ \left(1 - \dfrac{2a-(1-\alpha)}{2\alpha}\right)^2, & \text{if } a \in [1-\alpha, \alpha) \\ \left(\dfrac{(1-a)^2}{2\alpha(1-\alpha)}\right)^2, & \text{if } a \in [\alpha, 1] \end{cases} \tag{36}$$

- For $\alpha < \frac{1}{2}$:

$$\Pr\{\text{marry}|s=r,a\} = \begin{cases} \left(1 - \dfrac{a^2}{2\alpha(1-\alpha)}\right)^2, & \text{if } a \in [0, \alpha) \\ \left(1 - \dfrac{2a-\alpha}{2(1-\alpha)}\right)^2, & \text{if } a \in [\alpha, 1-\alpha) \\ \left(\dfrac{(1-a)^2}{2\alpha(1-\alpha)}\right)^2, & \text{if } a \in [1-\alpha, 1] \end{cases} \tag{37}$$

Proof of Proposition 2

We denote as $E[R_i^{t,s}|\text{marry}, a]$ the expectation of absolute rank of the potential partner j chosen by a player' i, under the condition that the marriage takes place in period t and $E[R_i^{t,s}|\text{marry}, a]$ is a function of a player i's strategy a. For $s = \bar{r}$, the conditional expectation is given by:

$$\begin{aligned} E[R_i^{t,\bar{r}}|\text{marry}, a] =& \frac{E[R_i^{t,\bar{r}}|R_i^{t,\bar{r}} > a] \Pr\{R_j^{t,\bar{r}} > a\}}{\Pr\{\text{marry}|s=\bar{r},a\}} = \frac{E[R_i^{t,\bar{r}}|R_i^{t,\bar{r}} > a] \Pr\{R_j^{t,\bar{r}} > a\}}{\Pr\{R_i^{t,\bar{r}} > a\} \Pr\{R_j^{t,\bar{r}} > a\}} \\ =& \frac{E[R_i^{t,\bar{r}}|R_i^{t,\bar{r}} > a]}{\Pr\{R_i^{t,\bar{r}} > a\}} = \frac{\int\limits_a^\infty u f_{R_i^{t,\bar{r}}}(u)du}{\int\limits_a^\infty f_{R_i^{t,\bar{r}}}(u)du} = \frac{\alpha I + 1 - \alpha + a}{2}, \end{aligned} \tag{38}$$

where $a = a(t, \bar{r}, I) \in [\alpha I, \alpha I + 1 - \alpha]$.

For $s = r$, we make use of the analysis carried out for determining the conditional expectation for $s = \bar{r}$ using equations (29), (30), (36), (37):

$$
\begin{aligned}
E[R_i^{t,r}|\text{marry}, a] =& \frac{E[R_i^{t,r}|R_i^{t,r} > a] \Pr\{R_j^{t,r} > a\}}{\Pr\{\text{marry}|s = r, a\}} \\
=& \frac{E[R_i^{t,r}|R_i^{t,r} > a]}{\Pr\{R_i^{t,r} > a\}} = \frac{\int\limits_a^\infty u f_{R_i^{t,r}}(u)du}{\int\limits_a^\infty f_{R_i^{t,r}}(u)du}.
\end{aligned}
\tag{39}
$$

For $\alpha \geq \frac{1}{2}$, equation (39) becomes:

$$
E[R_i^{t,r}|\text{marry}, a] = \begin{cases} \dfrac{2a^3 - 3\alpha(1-\alpha)}{3a^2 - 6\alpha(1-\alpha)}, & \text{if } a \in [0, 1-\alpha) \\ \dfrac{3a^2 - (1 + \alpha + \alpha^2)}{6a - 3(1+\alpha)}, & \text{if } a \in [1-\alpha, \alpha) \\ \dfrac{2a+1}{3}, & \text{if } a \in [\alpha, 1] \end{cases}
\tag{40}
$$

whereas for $\alpha < \frac{1}{2}$, equation (39) becomes:

$$
E[R_i^{t,r}|\text{marry}, a] = \begin{cases} \dfrac{2a^3 - 3\alpha(1-\alpha)}{3a^2 - 6\alpha(1-\alpha)}, & \text{if } a \in [0, \alpha) \\ \dfrac{3a^2 - (3 - 3\alpha + \alpha^2)}{6a - 3(2-\alpha)}, & \text{if } a \in [\alpha, 1-\alpha) \\ \dfrac{2a+1}{3}, & \text{if } a \in [1-\alpha, 1] \end{cases} \cdot
\tag{41}
$$

Proof of Proposition 3

To find the optimal strategy for period t and state $s = \bar{r}$ we first differentiate the expression in the right part of (11) with respect to a, then we equate the differential with zero and solve it for a. We denote the solution as $b^{t,\bar{r}}$. There are two solutions:

$$
\begin{aligned}
b_1^{t,\bar{r}} =& \frac{4}{3}\delta E^{t+1} - \frac{1}{3}\left[\alpha I + 1 - \alpha\right], \\
b_2^{t,\bar{r}} =& \alpha I + 1 - \alpha.
\end{aligned}
$$

Consider two possible cases for the value of expected rank E^{t+1}: $E^{t+1} < \frac{\alpha I + 1 - \alpha}{\delta}$ and $E^{t+1} \geq \frac{\alpha I + 1 - \alpha}{\delta}$.

1. Let $E^{t+1} < \frac{\alpha I + 1 - \alpha}{\delta}$, so that $b_1^{t,\bar{r}} < b_2^{t,\bar{r}}$. In this case the second derivative of $E^{t,\bar{r}}(a)$

with respect to a calculated in $b_1^{t,\bar{r}}$ ($b_2^{t,\bar{r}}$) equals to $\frac{-2(\alpha I+1-\alpha-\delta E^{t+1})}{(1-\alpha)^2}$ $\left(\frac{2(\alpha I+1-\alpha-\delta E^{t+1})}{(1-\alpha)^2}\right)$. Thus the strategy $a = b_1^{t,\bar{r}}$ maximizes $E^{t,\bar{r}}(a)$ whereas $a = b_2^{t,\bar{r}}$ minimizes $E^{t,\bar{r}}(a)$. Thus function (11) decreases in the interval $[b_1^{t,\bar{r}}, b_2^{t,\bar{r}}]$. If additionally $b_1^{t,\bar{r}} < \alpha I$, then the optimal strategy is the minimum possible value for the strategy, i.e. $a^*(N,\bar{r},I) = \alpha I$. For $b_1^{t,\bar{r}} \geq \alpha I$, then the strategy $a^*(N,\bar{r},I) = b_2^{t,\bar{r}}$ maximizes (11).

2. Let $E^{t+1} \geq \frac{\alpha I+1-\alpha}{\delta}$. In this case $b_2^{t,\bar{r}} < b_1^{t,\bar{r}}$ and $a = b_1^{t,\bar{r}}$ minimizes $E^{t,\bar{r}}(a)$ while $a = b_2^{t,\bar{r}}$ maximizes $E^{t,\bar{r}}(a)$. Function (11) increases from $a = \alpha I$ to $a = b_2^{t,\bar{r}}$ where obtains the maximum value.

6.1 Proof of Proposition 4

For brevity, we will consider the case $\alpha \geq \frac{1}{2}$ and omit the case $\alpha < \frac{1}{2}$ as it is very similar.[6] The problem is to find maximum of piece-wise function $E^{t,r}(a)$ with respect to strategy $a = a(t,r)$. This function is continuous with respect to a. When $a \in [0, 1-\alpha)$, then $E^{t,r}(a)$ has a unique maximum at $a = 0$. The second derivative of $E^{t,r}(a)$ calculated in $a = 0$ equals $\frac{4\delta E^{t+1}-1}{2\alpha(1-\alpha)}$. If $E^{t+1} < \frac{1}{4\delta}$, then the strategy $a^*(N,r,I) = 0$ maximizes $E^{t,r}(a)$. And at the same time $E^{t,r}(a)$ is a decreasing function with respect to parameter a in the interval of possible strategy values $[0,1]$. This means that the optimal strategy is $a^*(N,r,I) = 0$.

For $E^{t+1} \geq \frac{1}{4\delta}$, then $E^{t,r}(a)$ increases in the interval $a \in [0, 1-\alpha)$. Consider the case in which $a \in [1-\alpha, \alpha)$. Differentiation of $E^{t,r}(a)$ yields:

$$
\begin{aligned}
b_1^{t,r} =& \frac{1}{6}(1+\alpha) + \frac{2}{3}\delta E^{t+1} - \frac{1}{6}\sqrt{16\delta^2(E^{t+1})^2 - 16\delta E^{t+1}(1+\alpha) + 5\alpha^2 + 6\alpha + 5},\\
b_2^{t,r} =& \frac{1}{6}(1+\alpha) + \frac{2}{3}\delta E^{t+1} + \frac{1}{6}\sqrt{16\delta^2(E^{t+1})^2 - 16\delta E^{t+1}(1+\alpha) + 5\alpha^2 + 6\alpha + 5},
\end{aligned}
$$

where $b_1^{t,r} < b_2^{t,r}$. The second derivative of $E^{t,r}(a)$ in $b_1^{t,r}$ is negative, while the second derivative of $E^{t,r}(a)$ in $b_2^{t,r}$ is positive. Hence $b_1^{t,r}$ maximizes function $E^{t,r}(a)$ and $b_2^{t,r}$ minimizes it, and function $E^{t,r}(a)$ decreases from $b_1^{t,r}$ to $b_2^{t,r}$. Here we should consider three cases:

1. When $b_1^{t,r} < 1-\alpha$ which takes place if and only if $\frac{1}{4\delta} \leq E^{t+1} < \frac{5-19\alpha+11\alpha^2}{6(1-3\alpha)\delta}$, then $E^{t,r}(a)$ decreases on the interval $[1-\alpha, 1]$. Thus, the optimal strategy is $a^*(N,r,I) =$

[6] The complete proof can be provided upon request.

$1-\alpha$.

2. When $1-\alpha \leq b_1^{t,r} < \alpha$ (which takes place if and only if $\frac{5-19\alpha+11\alpha^2}{6(1-3\alpha)\delta} \leq E^{t+1} < \frac{5\alpha+1}{6\delta}$), then $E^{t,r}(a)$ increases on $[0, b_1^{t,r})$ and decreases on $(b_1^{t,r}, 1]$, so that the optimal strategy is $a = b_1^{t,r}$.

3. When $b_1^{t,r} \geq \alpha$, then $E^{t,r}(a)$ increases on $[0, \alpha)$. For $[\alpha, 1)$, $E^{t,r}(a)$ has one extreme point $a = b_3^{t,r}$, where
$$b_3^{t,r} = -\frac{1}{5} + \frac{6}{5}\delta E^{t+1},$$
and the second derivative shows that it maximizes $E^{t,r}(a)$. Hence $E^{t,r}(a)$ increases on $[0, b_3^{t,r})$ and decreases on $[b_3^{t,r}, 1]$, so that the optimal strategy is $a^*(N, r, I) = b_3^{t,r}$ if and only if $b_3^{t,r} \in [\alpha, 1)$, i.e. $E^{t+1} \geq \frac{5\alpha+1}{6\delta}$.

6.2 Proof of Proposition 6

First consider the case with assortative meeting, $s = \bar{r}$. For $a^*(t, \bar{r}, I) = \alpha I$, the payoff in equilibrium $E^{t,\bar{r}}(a^*(t, \bar{r}, I))$ is an increasing function of the universal rank I as

$$\frac{\partial E^{t,\bar{r}}(a^*)}{\partial I} = \alpha. \tag{42}$$

For $a^*(t, \bar{r}, I) = \alpha I + 1 - \alpha$, differentiation of $\partial E^{t,\bar{r}}(a^*)$ w.r.t. I yields:

$$\frac{\partial E^{t,\bar{r}}(a^*)}{\partial I} = \frac{\partial E^{t+1}}{\partial I}. \tag{43}$$

Given $\frac{\partial E^N}{\partial I} = \alpha\beta > 0$, we can easily prove the positiveness of $\frac{\partial E^N}{\partial I}$ for any $t = 1, \ldots, N-1$.

Consider next the case $a^*(t, \bar{r}, I) = \frac{4\delta E^{t+1} - (\alpha I + 1 - \alpha)}{3}$ when $\frac{\alpha I}{\delta} + \frac{1-\alpha}{4\delta} \leq E^{t+1} < \frac{\alpha I + 1 - \alpha}{\delta}$, in which:

$$\begin{aligned}\frac{\partial E^{t,\bar{r}}(a^*)}{\partial I} =& \frac{16\alpha}{9(1-\alpha)^2}\left(\delta E^{t+1} - (1-\alpha+\alpha I)\right)^2 \\ &- \frac{16\delta\left(\delta E^{t+1} - (1-\alpha+\alpha I)\right)^2 - 9\delta(1-\alpha)^2}{9(1-\alpha)^2}\frac{\partial E^{t+1}}{\partial I}.\end{aligned}$$

The right hand side is positive for any $t = 1, \ldots, N$ because $\frac{\alpha I}{\delta} + \frac{1-\alpha}{4\delta} \leq E^{t+1} < \frac{\alpha I + 1 - \alpha}{\delta}$ and the fact that $\frac{\partial E^N}{\partial I} = \alpha\beta > 0$. Therefore, in the assortative meeting case, a player's payoff in equilibrium is an increasing function of the universal rank I.

Consider then the case with random meeting, $s = r$, and suppose $\alpha \geq 1/2$ (the case where $\alpha < 1/2$ can be considered in the same way and leads to the same results). We show the proof when $a^* \in [0, 1-\alpha)$, and omit the cases in which $a^* \in [1-\alpha, \alpha)$ and $a^* \in [\alpha, 1]$, as the algebra is very similar and leads to the same results. We obtain the following expression of derivative:

$$\begin{aligned}\frac{\partial E^{t,r}(a^*)}{\partial I} =& 2\left(1 - \frac{(a^*)^2}{2\alpha(1-\alpha)}\right)\left(-\frac{a^* \frac{\partial a^*}{\partial I}}{\alpha(1-\alpha)}\right)\left(\frac{2(a^*)^3 - 3\alpha(1-\alpha)}{3(a^*)^2 - 6\alpha(1-\alpha)} - \delta E^{t+1}\right) \\ &+ \left(1 - \frac{(a^*)^2}{2\alpha(1-\alpha)}\right)^2 \left(\frac{18a^* \frac{\partial a^*}{\partial I}\alpha(1-\alpha)(1-2a^*)}{(3(a^*)^2 - 6\alpha(1-\alpha))^2} - \delta\frac{\partial E^{t+1}}{\partial I}\right) + \delta\frac{\partial E^{t+1}}{\partial I}.\end{aligned} \tag{44}$$

For $a^* = 0$ and $E^{t+1} < \frac{1}{4\delta}$, we obtain $E^{t,r}(a^*) = 1/2$, so that $E^{t,r}(a^*)$ is a non-decreasing function of parameter I. The similar result can be obtained for the case $a^* = 1-\alpha$ and $E^{t+1} \in [\frac{1}{4\delta}, \frac{5-19\alpha+11\alpha^2}{6(1-3\alpha)\delta})$. For $\frac{5-19\alpha+11\alpha^2}{6(1-3\alpha)\delta} \leq E^{t+1} < \frac{5\alpha+1}{6\delta}$ and $E^{t+1} \geq \frac{5\alpha+1}{6\delta}$ we obtain $\frac{\partial E^{t,r}(a^*)}{\partial I} \gtreqless 0$ if $\frac{\partial E^{t+1}}{\partial I} \gtreqless 0$. Given $\frac{\partial E^N}{\partial I} = \alpha\beta > 0$, we prove that for any $t = 1, \ldots, N$, the player's payoff in random meeting is an increasing function of I. Therefore, for any cases player's optimal payoff in random meetings is a non-decreasing function of the universal rank.

6.3 Proof of Proposition 7

It is straightforward that for $E^{t+1} < \frac{\alpha I}{\delta} + \frac{1-\alpha}{4\delta}$ and $E^{t+1} \geq \frac{\alpha I + 1 - \alpha}{\delta}$ the optimal strategy is a constant, so that $\frac{\partial a^*}{dI} = \alpha$ is a non-decreasing function of the universal rank I.

Consider next $\frac{\alpha I}{\delta} + \frac{1-\alpha}{4\delta} \leq E^{t+1} < \frac{\alpha I + 1 - \alpha}{\delta}$, for which the optimal strategy is $a^*(t, \bar{r}, I) = \frac{4\delta E^{t+1} - (\alpha I + 1 - \alpha)}{3}$. Here we obtain $\frac{\partial a^*}{dI} = \frac{4\delta}{3}\frac{\partial E^{t+1}}{\partial I} - \frac{\alpha}{3}$, where $\frac{\partial E^{t+1}}{\partial I} = \beta\frac{\partial E^{t+1,\bar{r}}}{\partial I} + (1-\beta)\frac{\partial E^{t+1,r}}{\partial I}$. Thus $\frac{\partial a^*}{dI}$ is non-negative during the whole game if and only if $\frac{\partial E^{t+1}}{\partial I} > \frac{\alpha}{4\delta}$.

For $t = N-1$, the player's payoff $\frac{\partial E^N}{\partial I} = \alpha\beta$, hence the condition of non-negativity is $\beta > \frac{1}{4\delta}$. Now consider $t = N-2$. Substituting the optimal strategy $a^*(N-2, \bar{r}, I)$ into expression (44) and writing down the condition $\frac{\partial E^{N-1}}{\partial I} > \frac{\alpha}{4\delta}$ yields:

$$\left[\delta E^{t+1} - (1 - \alpha + \alpha I)\right]^2 \frac{16}{9(1-\alpha)^2}(1 - \delta\beta)\,\delta + \left(\delta^2\beta - \frac{1}{4}\right) > 0.$$

Given $\beta > \frac{1}{4\delta^2}$ we can easily prove that $\frac{\partial E^{N-1}}{\partial I} > \frac{\alpha}{4\delta}$. Therefore $a^*(N-2, \bar{r}, I)$ is a non-decreasing function of universal rank I. By repeating the procedure recurrently for all t

we prove the result of the proposition.

6.4 Proof of Proposition 8

Consider $\alpha \geq 1/2$ (for brevity we omit the calculations for $\alpha < 1/2$ as they are the same). The non-negativity of $\partial a^*(t,r,I)/\partial I$ is straightforward for $E^{t+1} < \frac{5-19\alpha+11\alpha^2}{6(1-3\alpha)\delta}$. The optimal strategy $a^*(t,r,I)$ is a non-decreasing function for $E^{t+1} \geq \frac{5\alpha+1}{6\delta}$ iff $\partial E^{t+1}/\partial I$ is non-negative, which is proved by Proposition 6.

Now consider $\frac{5-19\alpha+11\alpha^2}{6(1-3\alpha)\delta} \leq E^{t+1} < \frac{5\alpha+1}{6\delta}$:

$$\frac{\partial a^*(t,r,I)}{\partial I} = \frac{2\delta}{3}\frac{\partial E^{t+1}}{\partial I}\left(1 - \frac{4\delta E^{t+1} - 2(1-\alpha)}{\sqrt{16\delta^2\left(E^{t+1}\right)^2 - 16\delta E^{t+1}\left(1-\alpha\right) + 5\alpha^2 + 6\alpha + 5}}\right).$$

And we can easily obtain $\frac{\partial a^*(t,r,I)}{\partial I} \gtreqless 0$ when $\frac{\partial E^{t+1}}{\partial I} \gtreqless 0$ since the right hand side is always positive when $\frac{5-19\alpha+11\alpha^2}{6(1-3\alpha)\delta} \leq E^{t+1} < \frac{5\alpha+1}{6\delta}$. Therefore, given Proposition 6 we prove Proposition 8.

Alma Mater Studiorum - Università di Bologna
DEPARTMENT OF ECONOMICS

Strada Maggiore 45
40125 Bologna - Italy
Tel. +39 051 2092604
Fax +39 051 2092664
http://www.dse.unibo.it