RELIABILITY AND PERFORMANCES OF FINITE ELEMENT CAD TOOLS FOR THE SOLUTION OF MICROWAVE PROBLEMS

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ABSTRACT

The target of this paper is two-fold. On the one hand, we address the question of the reliability of' many recently proposed high-order finite elements. On the other hand, new quadrilateral edge elements are proposed and the improvements of the performances they permit are stressed.

INTRODUCTION

High-order edge elements make it possible to improve the performances of finite element simulators. For this reason, in recent years, quite a lot of high-order edge elements have been introduced [1, 2, 3] after the pioneering work of Nedelec [4]. However, before analysing their performances, one should establish the reliability of the finite elements considered. This is an important issue since, for many years, finite element simulators were plagued by the so-called spurious modes.

Thus, in this paper, we firstly report some recent results of the present authors assessing the reliability of many of the new elements recently introduced. Once the class of a priori reliable finite elements has sufficiently been widened, it makes sense to compare the performances in order to find the best one. However, the "best" element could also be an element not defined so far and, for this reason, it is important to keep on looking for new and more efficient elements. As a first attempt in this direction, we will firstly define a new element on rectangles and then we will compare its performances with those of other well known spurious-free elements. In the simple example considered the performances of the new element are significantly better than those of all existing elements of the same order.

HIGH-ORDER SPURIOUS-FREE FINITE ELEMENT APPROXIMATIONS

It is now well established that a finite element technique is spurious-free if and only if the corresponding finite element space satisfies the three necessary and sufficient conditions ((CAS), (CDK) and (DCP)) reported in [5]. Moreover, it is now known that all elements of both Nedelec's families defined on triangles or tetrahedral guarantee spurious-free approximations [6]. This latter result implies that nowadays all problems of interest can be reliably solved by using the finite element method when it is based on Nedelec's elements. However, this does not exclude that the introduction of new spurious-free elements could lead to an improvement of the performances. For this reason many researchers recently proposed several new elements. Due to the limited space available, let us just report some recent results of the present authors which widen significantly the class of a priori reliable finite elements. It is proved in [7] that the elements proposed by Kameari [2], Ahagon and Kashimoto [8], Yioultsis and Tsiboukis [9], Peterson et al. [10, 3] and Lee et al. [1] are spurious-free.

NEW SPURIOUS-FREE RECTANGULAR ELEMENTS AND THEIR PERFORMANCES

Let us consider the cross-section of a hollow rectangular waveguide and the corresponding problem of finding the transverse magnetic field of TM modes at cutoff.

This simple problem can be approximated by using a family of uniform meshes made up of rectangular elements. In this case we already know that all rectangular Nedelec's edge elements of the first family provide spurious free approximations [11].

In this section we will use notations similar to those introduced in [12] to denote the different finite elements considered. For instance $Q_{i,j}$ will denote the finite element, introduced in [4], [13], whose vector basis functions belong to $Q_{i,j} \times Q_{j,i}$, where $Q_{p,q}$ is the space of polynomials in two variables x, y, having maximum degree equal to p in x and to q in y. Different superscripts will be used to denote elements built starting from these standard finite elements. Moreover, let $V = H(curl, \Omega) = \{\vec{v} | \vec{v}, \nabla \times \vec{v} \in L^2(\Omega)^2\}$, $V_0 = \{\vec{v} \in V | \nabla \times \vec{v} = 0\}$ and let V_1 be the orthogonal complement of V_0 in V.

In order to widen the class of spurious free rectangular elements let us consider a generic rectangular element $K = \{(x, y) \in R^2 \mid y_1 \le y \le y_2, x_1 \le x \le x_2\}$ (with $y_1 < y_2$ and $x_1 < x_2$) and the following basis functions defining a new element $Q^s_{1,1}$:

$$\vec{w}_1 = \vec{e}_x \frac{y - y_2}{y_1 - y_2}, \quad \vec{w}_2 = \vec{e}_x \frac{2x - x_1 - x_2}{x_2 - x_1} \frac{y - y_2}{y_1 - y_2} + \vec{e}_y \frac{(x - x_1)(x_2 - x)}{(x_2 - x_1)(y_2 - y_1)}$$

$$\vec{w}_3 = \vec{e}_x \frac{y - y_1}{y_2 - y_1}, \quad \vec{w}_4 = \vec{e}_x \frac{2x - x_1 - x_2}{x_2 - x_1} \frac{y - y_1}{y_2 - y_1} - \vec{e}_y \frac{(x - x_1)(x_2 - x)}{(x_2 - x_1)(y_2 - y_1)}$$

$$\vec{w}_5 = \vec{e}_y \frac{x - x_2}{x_1 - x_2}, \quad \vec{w}_6 = \vec{e}_y \frac{2y - y_1 - y_2}{y_2 - y_1} \frac{x - x_2}{x_1 - x_2} + \vec{e}_x \frac{(y - y_1)(y_2 - y)}{(x_2 - x_1)(y_2 - y_1)}$$

$$\vec{w}_7 = \vec{e}_y \frac{x - x_1}{x_2 - x_1}, \quad \vec{w}_8 = \vec{e}_y \frac{2y - y_1 - y_2}{y_2 - y_1} \frac{x - x_1}{x_2 - x_1} - \vec{e}_x \frac{(y - y_1)(y_2 - y)}{(x_2 - x_1)(y_2 - y_1)}$$

where $\vec{e}_x$ and $\vec{e}_y$ are the coordinate unit vectors. Note that each basis function has a nonzero tangential component along one edge only: $\vec{w}_1$ and $\vec{w}_2$ along the edge $y = y_1$, $\vec{w}_3$ and $\vec{w}_4$ along $y = y_2$, $\vec{w}_5$ and $\vec{w}_6$ along $x = x_1$ and $\vec{w}_7$ and $\vec{w}_8$ along $x = x_2$. Note also that the two basis functions associated to the same edge interpolate the tangential component with different polynomial orders. This means that in order to define a *curl* conforming approximation each basis function must be matched with only the corresponding basis function of the adjacent element. Then, in particular, we can write the finite element space V_h as $V_h = U_h \oplus W_h$ (see [6] for many analogous decompositions of V_h), where U_h is the finite element space generated by the basis functions $\vec{w}_1$, $\vec{w}_3$, $\vec{w}_5$ and $\vec{w}_7$ (i.e., the same space generated by $Q_{0,1}$) and W_h, being the space generated by the basis functions $\vec{w}_2$, $\vec{w}_4$, $\vec{w}_6$ and $\vec{w}_8$ is such that $W_h \subset V_0$. As already noted the spaces generated by $Q_{0,1}$ satisfy (CAS), (CDK) and (DCP). Thus by Lemmas 25, 26 and 27 of [6] we immediately conclude that the spaces V_h generated by the new element $Q^s_{1,1}$ can be used to obtain spurious-free approximations.

Note that $Q^s_{1,1}$ can be thought as a "stabilized $Q_{1,1}$ element". In fact, if the second terms are dropped in the expressions of $\vec{w}_2$, $\vec{w}_4$, $\vec{w}_6$, and $\vec{w}_8$, the new $Q^s_{1,1}$ reduces to $Q_{1,1}$, which is known to give spurious modes [12]. On the other hand, the second terms of $\vec{w}_2$, $\vec{w}_4$, $\vec{w}_6$, and $\vec{w}_8$ serve only to make curl-free the corresponding basis function, since their tangential components vanish on the whole boundary of *K*.

In order to define other finite element spaces providing spurious free approximations let us consider the following basis functions having zero tangential components on the boundary of *K:*

$$\vec{w}_9 = \vec{e}_x \frac{2x - x_1 - x_2}{(x_1 - x_2)(x_2 - x_1)} \frac{(y - y_2)(y - y_1)}{(y_1 - y_2)(y_2 - y_1)} + \vec{e}_y \frac{2y - y_1 - y_2}{(y_1 - y_2)(y_2 - y_1)} \frac{(x - x_2)(x - x_1)}{(x_1 - x_2)(x_2 - x_1)}$$

$$\vec{w}_{10} = \vec{e}_x (y - y_1)(y_2 - y), \quad \vec{w}_{11} = \vec{e}_y (x - x_1)(x_2 - x)$$

$$\vec{w}_{12} = \vec{e}_x \frac{2x - x_1 - x_2}{(x_1 - x_2)(x_2 - x_1)} \frac{(y - y_2)(y - y_1)}{(y_1 - y_2)(y_2 - y_1)} - \vec{e}_y \frac{2y - y_1 - y_2}{(y_1 - y_2)(y_2 - y_1)} \frac{(x - x_2)(x - x_1)}{(x_1 - x_2)(x_2 - x_1)}$$

On the generic rectangle *K* let us consider as basis functions all the functions $\vec{w}_1, \dots, \vec{w}_{12}$ so far considered. In this case we obtain the well-known $Q_{1,2}$ element, as $\vec{w}_1, \dots, \vec{w}_8, \vec{w}_{10}, \vec{w}_{11}$ may be used to generate all bilinear vector functions belonging to $Q_{1,1}$ [12] and $\vec{w}_9, \vec{w}_{10}, \vec{w}_{11}$ and $\vec{w}_{12}$ are just the basis functions belonging to $Q_{1,2}$ and not to $Q_{1,1}$.

Other choices of the basis functions are possible, however. As a matter of fact consider the three sets of basis functions $\{\vec{w}_1, \dots, \vec{w}_8, \vec{w}_{10}, \vec{w}_{11}, \vec{w}_{12}\}$, $\{\vec{w}_1, \dots, \vec{w}_8, \vec{w}_9, \vec{w}_{10}, \vec{w}_{11}\}$ and $\{\vec{w}_1, \dots, \vec{w}_8, \vec{w}_{10}, \vec{w}_{11}\}$ defining, respectively, three different elements denoted by $Q^{-w_9}_{1,2}$, $Q^{-w_{12}}_{1,2}$ and $Q^{-w_9-w_{12}}_{1,2}$.

By considering that the spaces V_h generated by these three elements contain the space U_h generated by $Q_{0,1}$ and that the spaces U_h satisfy (CAS) and (CDK) [4] we immediately conclude that all these spaces V_h satisfy (CAS) and (CDK) (see Lemma 25 and 26 of [6]).

The proof that they also satisfy (DCP) is only sketched here. First of all note that $\nabla \times \vec{w}_{10}$ is a first order polynomial depending only on *y*, that $\nabla \times \vec{w}_{11}$ is a first order polynomial depending only on *x* and that $\nabla \times \vec{w}_{12}$ is a second order polynomial. Thus, the finite dimensional subspace *Z* generated by $\vec{w}_{10}$, $\vec{w}_{11}$ and $\vec{w}_{12}$ on a single element satisfy the hypotheses of Lemma 30 of [6] and then $\|\vec{z}\|_{1,K} \leq C \|\vec{z}\|_{V,K}$ $\forall \vec{z} \in Z$ where the norms are those of the spaces $H^1(K)^2$ and $H(curl, K)$, respectively. Now the conclusion follows by noting that $V_1 \subset H^1(\Omega)^2$, where $\Omega \subset R^2$ is the domain representing the waveguide cross section, and by applying a much simplified version of Lemma 36 of [6].

This proves that the spaces V_h generated by $Q^{-w_9}_{1,2}$ and $Q^{-w_9-w_{12}}_{1,2}$ provide spurious free approximations. To obtain the same results also for the spaces *Vh* generated by $Q^{-w_{12}}_{1,2}$ note that they can be obtained from the spaces U_h generated by $Q^{-w_9-w_{12}}_{1,2}$ by adding one more basis function $\vec{w}_9$ to any single element. As $\nabla \times \vec{w}_9 = 0$ we have that we can think of V_h as $V_h = U_h + W_h$ with $W_h \subset V_0$ and with U_h satisfying (CDK) and (DCP). Then by Lemma 27 of [6] we have that also the spaces V_h satisfy it.

It is now meaningful to carry out an investigation on the performances of the elements introduced above. This is done by considering again the simple 2D problem introduced above. In the limited space of this paper we address just the question of the performances of the new element $Q^{-w_9}_{1,2}$ which resulted to be the best one.

One could note that the same order of approximation as that of $Q_{1,2}$ is possible by using $Q^{-w_9}_{1,2}$. Moreover the image of $Q_{1,2}$ by the *curl* operator is exactly the same as that of $Q^{-w_9}_{1,2}$ since $\vec{w}_9$ is irrotational. Thus we expect the latter to provide the same order of approximation as the former but with fewer degrees of freedom. In this sense we expect better performances of the new element $Q^{-w_9}_{1,2}$ and this expectation will be confirmed by the following numerical analysis.

It is assumed that the hollow rectangular waveguide is 2 cm wide and 1 cm high. This problem admits the analytical evaluation of all eigenvalues $\omega_i,\ i = 1, 2, \ldots, \infty$. We will calculate the first ten numerical eigenvalues $w_{i,h}$, $i = 1, \ldots, 10$ by using different elements. The accuracy of the approximate solution will be assessed by the value of $e = (1/10)\sum_{i=1}^{10} (|\omega_i - \omega_{i,h}| \,/\, |\omega_i|)$.

However, in order to compare the performances of different elements in terms of their computational efficiency we have to estimate the computational cost needed to achieve a given accuracy. Good estimates can be the number, n_{dofs}, of degrees of freedom used in the finite element simulation and the number, $n_{\neq 0}$, of nonzero entries of the finite element matrices defining the generalized algebraic eigenvalue problem from which the approximate solution is calculated. The first parameter n_{dofs} is well suited when the algebraic eigenvalue problem is tackled by using subroutines which store the finite element matrices as band matrices, whereas the second parameter $n_{\neq 0}$ is much more significant when the same problem is tackled by using subroutines based on iterative techniques which store the finite element matrices as sparse matrices.

Our target is to compare the performances of rectangular elements. However, in order to know if these elements are meaningful at least for particular geometries we will also consider the performances obtained by using the spaces generated by low order Nedelec's edge elements defined on triangles. In particular we will consider the first three triangular elements of the first family [4] (denoted by R_1, R_2 and R_3).

In Figure 1 we show the behaviour of e as a function of n_{dofs} when different elements are used. As can be seen the absolute performances of $Q_{1,2}$ and $Q_{1,2}^{-w_9}$ are good since they provide better approximations than the corresponding incomplete second order elements defined on triangles (R_2). As expected the best performances are obtained by using R_3 since this is an element of higher order. Analogously, the lowest order elements R_1 and $Q_{0,1}$ provide the worst performances. By focusing our analysis on rectangular elements only and on $Q_{1,2}$ and $Q_{1,2}^{-w_9}$ in particular, we may note that the performances of $Q_{1,2}^{-w_9}$ are better than those of $Q_{1,2}$.

Figure 2 shows the behaviour of e versus $n_{\neq 0}$. In this case the rectangular elements $Q_{1,2}$ and $Q_{1,2}^{-w_9}$ provide excellent absolute performances ($Q_{1,2}^{-w_9}$ is even better than R_3 at least in the range of $n_{\neq 0}$ here considered). Moreover, if we again focus our attention on rectangular elements only, we note that the relative performances are the same as those shown in Figure 1.

Note that by using $Q_{1,2}^{-w_9}$ instead of Q_{12} it is possible to obtain a given error with a significant reduction of n_{dofs} ($\simeq 12\%$ in all simulations) or $n_{\neq 0}$ ($\simeq$ 19% in all simulations). The impact of these reductions on the CPU times necessary to solve the generalized algebraic eigenproblems is even more significant. As a matter of fact, the reductions obtained by using the Arnoldi Package software [14] are always in the range 31% ÷ 42%.

References

[1] J. F. Lee, D. K. Sun, and Z. J. Cendes, "Tangential vector finite elements for electromagnetic field computation," *IEEE Transactions on Magnetics,* vol. 27, no. 5, pp. 4032-4035, 1991.

[2] A. Kameari, "Symmetric second order edge elements for triangles and tetrahedral," *IEEE Transactions on Magnetics,* vol. 35, pp. 1394–1397, May 1999.

[3] R. D. Graglia, D. R. Wilton, and A. F. Peterson, "Higher order interpolator vector bases for computational electromagnetics," *IEEE Transactions on Antennas and Propagation,* vol. 45, pp. 329–342, March 1997.

[4] J. C. Nedelec, "Mixed finite elements in R^3," Numerische Mathematik, vol. 35, pp. 315–341, 1980.

[5] S. Caorsi, P. Fernandes, and M. Raffetto, "On the convergence of Galerkin finite element approximations of electromagnetic eigenproblems," *SIAM Journal on Numerical Analysis,* vol. 38, no. 2, pp. 580–607, 2000.

[6] S. Caorsi, P. Fernandes, and M. Raffetto, "Spurious-free approximations of electromagnetic eigenproblems by means of Nedelec-type elements," *Mathematical Modelling and Numerical Analysis,* vol. 35, no. 2, pp. 331–354, 2001.

[7] P. Fernandes and M. Raffetto, "Characterization of spurious-free finite element methods in electromagnetics," *COMPEL,* vol. 21, no. 1, 2002. In press.

[8] A. Ahagon and T. Kashimoto, "Three-dimensional electromagnetic wave analysis using high order edge elements," *IEEE Transactions on Magnetics,* vol. 31, pp. 1753-1756, May 1995.

[9] T. V. Yioultsis and T. D. Tsiboukis, "Development and implementation of second and third order vector finite elements in various 3-d electromagnetic field problems," *IEEE Transactions on Magnetics*, vol. 33, pp. 1812–1815, March 1997.

[10] J. S. Savage and A. F. Peterson, "Higher-order vector finite elements for tetrahedral cells electromagnetics," *IEEE Transactions on Microwave Theory and Techniques,* vol. 44, pp. 874–879, June 1996.

[11] F. Kikuchi, M. Yamamoto, and H. Fujio, "Theoretical and computational aspects of Nedelec's edge elements for electromagnetic," *Computational Mechanics - New trends and applications, Eds.: E. Onate and S. R. Idelsohn, CIMNE, Barcelona, Spain,* pp. 1–16, 1998.

[12] B. M. Dillon, P. T. S. Liu, and J. P. Webb, "Spurious modes in quadrilateral and triangular edge elements," *COMPEL,* vol. 13–Supplement A, pp. 311–316, May 1994.

[13] J. C. Nedelec, "A new family of mixed finite elements in R^3," *Numerische Mathematik,* vol. 50, pp. 57–81, 1986.

[14] R. B. Lehoucq, D. C. Sorensen, and C. Yang, *ARPACK Users' Guide: Solution of Large-Scale Eigenvalue Problems with Implicitly Restarted Arnoldi Methods.* Philadelphia, PA: SIAM, 1998.

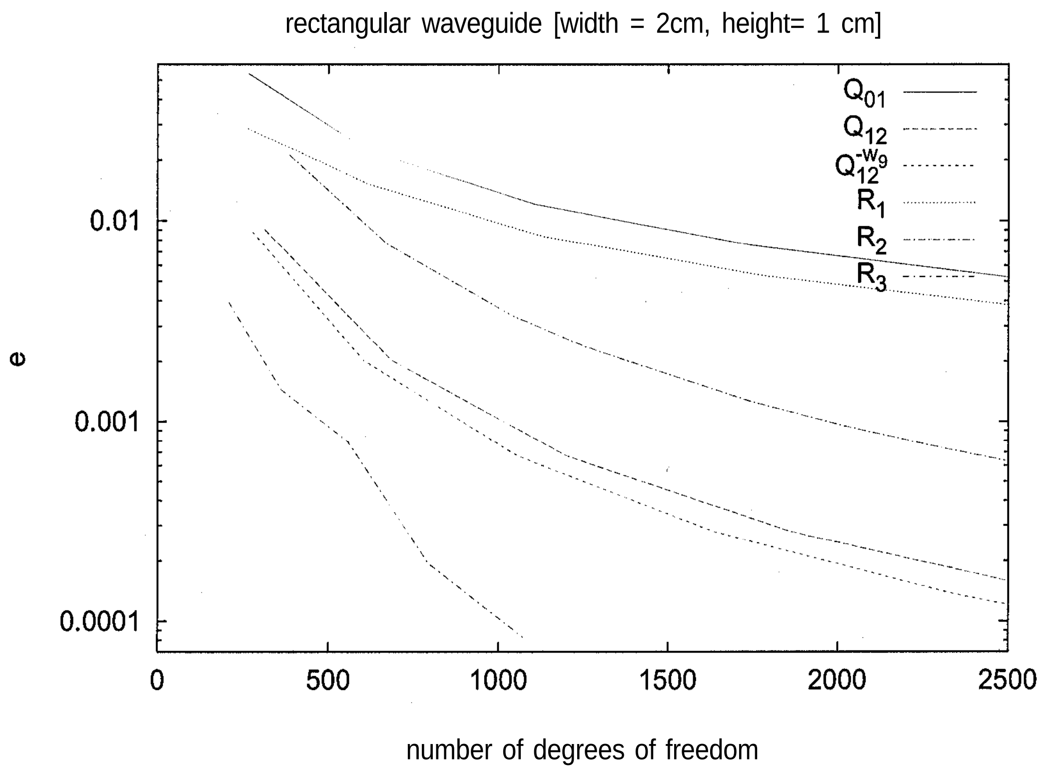

Figure 1: Performances of different finite elements in terms of e versus n_{dofs}

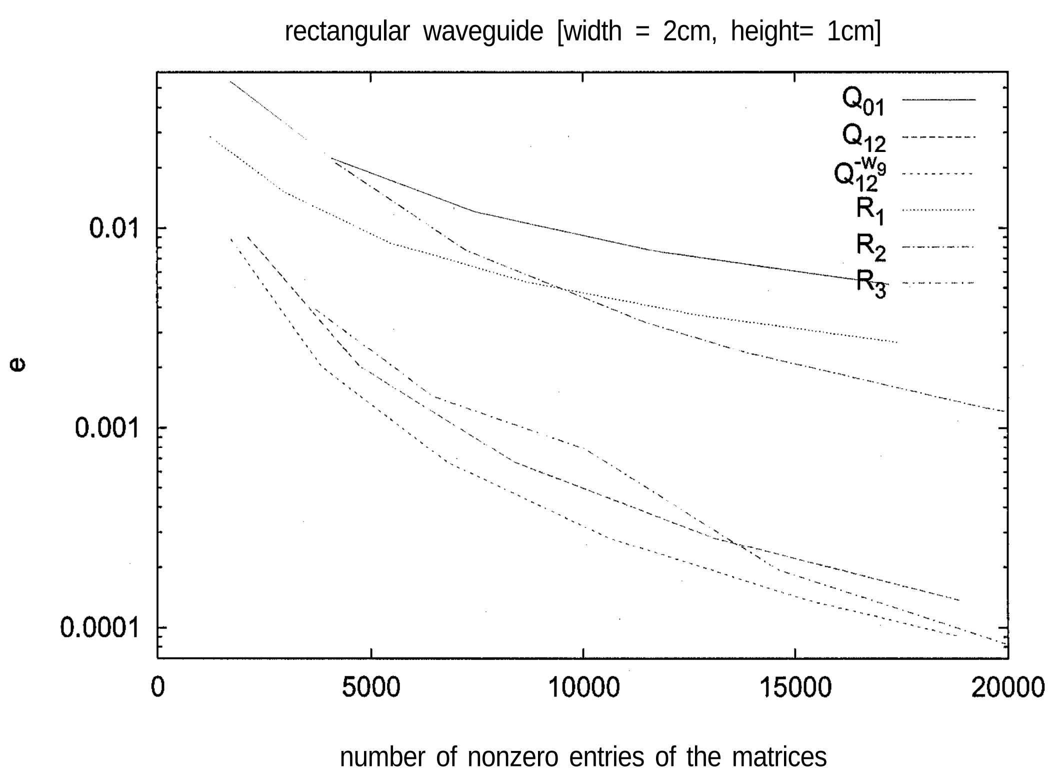

Figure 2: Performances of different finite elements in terms of e versus $n_{\neq 0}$