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On the Feedback Solution of a Differential Oligopoly Game with Hyperbolic Demand and Capacity Accumulation

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On the Feedback Solution of a Differential Oligopoly Game with Hyperbolic Demand and Capacity Accumulation[1]

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Abstract

I characterise the subgame perfect equilibrium of a differential market game with hyperbolic demand where firms are quantity-setters and accumulate capacity over time *à la* Ramsey. I show that the open-loop solution is subgame perfect. Then, I analyse the feasibility of horizontal mergers, and compare the result generated by the dynamic setup with the merger incentive associated with the static model. It appears that allowing for the role of time makes mergers more likely to occur than they would on the basis of the static setting.

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1 Introduction

Most of the existing literature on oligopoly theory (either static or dynamic) assumes linear demand functions, as this, in addition to simplifying calculations, also ensures both concavity and unicity of the equilibrium, which, in general, wouldn't be warranted in presence of convex demand systems (see Friedman, 1977; and Dixit, 1986, *inter alia*). However, the use of linear demand function is in sharp contrast with the standard microeconomic approach to consumer behavior, where the widespread adoption of Cobb-Douglas preferences (or their log-linear affine transformation) yields hyperbolic demand functions. The same applies to the so-called quasi-linear utility function, concave in consumption and linear in money, that again yields a convex demand system. Indeed, both preference structures share the common property of producing isoelastic demand functions.[1] In fact, this is sometimes openly referred to in the field of industrial organization, where researchers mentions the opportunity of dealing with non-linear demand functions, and then promptly leave it aside for the sake of tractability.[2] Additionally, the econometric approach to demand theory has produced the highest efforts to building up a robust approach to the estimation of non-linear individual and market demand functions, yielding a large empirical evidence in this direction.[3] With these considerations in mind, it appears desirable to investigate the bearings of non-linear demand systems on the performance of firms operating in oligopolistic markets, using thus a setup with solid microfoundations corroborated by robust empirical evidence, even though this is a costly approach in terms of analytical tractability.

[1] For a thourough illustration of these issues in consumer theory, see the classical textbooks of Deaton and Muellbauer (1980), and Varian (1992), *inter alia*.

[2] A noteworthy example in this respect is Shy (1995, pp. 53-54), using quasi-linear utility function to define the concept of consumer surplus.

[3] See Hausman (1981) and Varian (1982, 1990), *inter alia*.

With specific reference to differential games, the use of linear demand functions (jointly with either linear or quadratic cost functions) allows for the closed-form solution of the feedback equilibrium through the Bellman equation of the representative firm, as the model takes a linear-quadratic form and therefore one can stipulate that the corresponding candidate value function is also linear-quadratic. However, there is no particular reason to believe that a linear function describes correctly virtually any market demand in the real world, and therefore it is of primary interest to design, if possible, closed-form solutions of market games with non-linear demand functions.[4]

The aim of this paper is to illustrate a way out of the aforementioned problem, offered by dynamic game theory. I illustrate a dynamic Cournot model where firms (i) accumulate capacity *à la* Ramsey (1928), (ii) bear an instantaneous cost of holding any given capacity, and (iii) discount future profits at a constant rate. The main results are threefold. First, I show that the resulting open-loop equilibrium is indeed subgame perfect as it is a (degenerate) feedback equilibrium. Secondly, a straightforward feature of the equilibrium is that - unlike the static game - it admits an economically sensible solution even in the limit case where the marginal production cost of the consumption good drops to zero. This is entirely due to the dynamic nature of this setup. Finally, I use it to investigate the profit (or, private) incentive towards horizontal mergers, to find that taking a dynamic perspective widens the range of privately feasible mergers.[5] That is, the presence of

[4] To the best of my knowledge, the only existing examples of differential oligopoly games with non-linear market demand are in Cellini and Lambertini (2007) and Lambertini (2010). The first uses a non-linear demand *à la* Anderson and Engers (1992) and also investigates horizontal mergers. The second uses a hyperbolic demand with sticky prices (as in Simaan and Takayama, 1978; and Fershtman and Kamien, 1987), but leaves the merger issue out of the picture.

[5] To the best of my knoweldge, scanty attention has been devoted to the implications of

discounting, depreciation and a cost associated to holding capacity increases the firms' willingness to merge horizontally as compared to the static setup, for any admissible merger size. Any merger, of course, has undesirable consequences on consumer surplus and ultimately for welfare (at least in this model, where the efficiency defense is not operating).

The remainder of the paper is structured as follows. the static game is briefly summed up in section 2. Section 3 laids out the dynamic setting. The Cournot-Ramsey game is solved in section 3, while the profitability of horizontal mergers is investigated in section 4. Section 5 concludes.

2 A summary of the static game

Consider a market where N single-product firms supply individual quantities q_i, $i = 1, 2, 3, ...N$. The good is homogeneous, and market demand is $p = a/Q$, $Q = \sum_{i=1}^{N} q_i$. This demand function is the outcome of the constrained maximum problem of a representative consumer endowed with a log-linear utility function

$$U = Log\,[Q] + m \tag{2.1}$$

where m is a numeraire good whose price is normalised to one. The budget constraint establishes that the consumer's nominal income Y must be large enough to cover the expenditure, so that $Y \geq pQ + m$. The representative consumer must

$$\max_{Q} L = U + \mu\,(Y - pQ - m)\,. \tag{2.2}$$

dynamic competition on merger incentives, with the exceptions of Dockner and Gaunersdorfer (2001) and Benchekroun (2003), using a price dynamics à *la* Simaan and Takayama (1978) and Cellini and Lambertini (2007) adopting a Ramsey-type capital accumulation dynamics. All of these contrivutions, however, assume linear demand functions.

Solving the above problem, one obtains indeed the hyperbolic demand function $p = a/Q$.

On the supply side, production entails a total cost $C_i = cq_i$, where $c > 0$ is a constant parameter measuring marginal production cost. Market competition takes place *à la* Cournot-Nash; therefore, firm i chooses q_i so as to maximise profits $\pi_i = (p - c)\, q_i$. This entails that the following first order condition must be satisfied (assuming interior solutions):

$$\frac{\partial \pi_i}{\partial q_i} = \frac{aQ_{-i}}{(q_i + Q_{-i})^2} - c = 0 \tag{2.3}$$

where $Q_{-i} \equiv \sum_{j \neq i} q_j$. The associated second order condition:

$$\frac{\partial^2 \pi_i}{\partial q_i^2} = -\frac{2a \sum_{j \neq} q_j}{(q_i + Q_{-i})^3} \leq 0 \tag{2.4}$$

is always met. then, imposing the symmetry condition $q_i = q$ for all $q_i = 1, 2, 3, ...N$, one obtains the individual Cournot-Nash equilibrium output $q^{CN} = a\,(n-1) / (N^2 c)$, yielding profits $\pi^{CN} = a/N^2$. If the N firms were operating under perfect competition, then $p^* = c$ and therefore $q^* = a/(Nc)$.

It is apparent that the above solutions (i.e., both the Cournot-Nash equilibrium and the perfectly competitive equilibrium) are determinate for all $c > 0$, while they become indeterminate in correspondence of $c = 0$.

Now I will turn my attention to a differential game where demand, cost, and profits are the same as here but firms accumulate productive capacity in a Ramsey fashion.

3 The dynamic setup

The market exists over $t \in [0, \infty)$, and is served by N firms producing a homogeneous good. Let $q_i(t)$ define the quantity sold by firm i at time t.

Firms compete *à la* Cournot, the demand function at time t being:

$$p(t) = \frac{a}{Q(t)}, \; Q(t) = \sum_{i=1}^{N} q_i(t); \; a > 0. \tag{3.1}$$

In order to produce, firms bear quadratic instantaneous costs $C_i(t) = cq_i(t)$. Moreover, they must accumulate capacity or physical capital $k_i(t)$ over time. The two models I consider in the present paper are characterised by two different kinematic equations for capital accumulation as in Ramsey (1928), with the following dynamic equation:

$$\frac{dk_i(t)}{dt} \equiv \dot{k}_i = Ak_i(t) - q_i(t) - \delta k_i(t), \tag{3.2}$$

where $Ak_i(t) = y_i(t)$ denotes the output produced by firm i at time t. I.e., this is the familiar $A-k$ version of the Ramsey model. Capital accumulates as a result of intertemporal relocation of unsold output $y_i(t) - q_i(t)$.[6] This can be interpreted in two ways. The first consists in viewing this setup as a corn-corn model, where unsold output is reintroduced in the production process. The second consists in thinking of a two-sector economy where there exists an industry producing the capital input which can be traded against the final good at a price equal to one (for further discussion, see Cellini and Lambertini, 2007). Unlike the standard macroeconomic approach to the Ramsey growth model, here I will allow for the presence of an instantaneous cost of holding installed capacity. This cost will be $\Gamma_i(t) = bk_i(t)$, with $b \geq 0$. In the remainder, I will refer to b as a measure of the *opportunity cost* of a unit of capacity. The control variable is $q_i(t)$, while the state variable is $k_i(t)$.

Assuming all firms discount profits at the same constant rate $\rho \geq 0$, the problem of firm i is to choose the output level $q_i(t)$ so as to maximisize its

[6]Of course, capacity decumulates whenever $y_i(t) - q_i(t) \leq 0$.

own discounted profits:

$$\Pi_i(\mathbf{k}(t), \mathbf{q}(t)) \triangleq \left\{ \int_0^\infty [p(t) - c] q_i(t) - b k_i(t) \right\} e^{-\rho t} dt \tag{3.3}$$

s.t. the price dynamics (3.2) and the initial conditions $k_i(0) = k_{i0}$. $\mathbf{k}(t)$ and $\mathbf{q}(t)$ are the vector of all firms' states and controls, respectively. In order to make the model consistent with the corresponding macroeconomic approach, I will set $A > \delta + \rho$.

4 The Cournot-Ramsey game

Here I will illustrate the open-loop solution of a generic firm in the industry. The Hamiltonian of firm i is:

$$\begin{aligned} \mathcal{H}_i(\mathbf{k}(t), \mathbf{q}(t)) &= e^{-\rho t} \left\{ \frac{a q_i(t)}{q_i(t) + \sum_{j \neq i} q_j(t)} - c q_i(t) - b k_i(t) \right. \\ &\quad + \lambda_{ii}(t) [A k_i(t) - q_i(t) - \delta k_i(t)] \\ &\quad \left. + \sum_{j \neq i} \lambda_{ij}(t) [A k_j(t) - q_j(t) - \delta k_j(t)] \right\} \end{aligned} \tag{4.1}$$

where $\lambda_{ij}(t) = \mu_{ij}(t) e^{\rho t}$, $\mu_{ij}(t)$ being the co-state variable that firm i associates to $k_{ij}(t)$.

The first order condition on control $q_i(t)$ is:[7]

$$\frac{\partial \mathcal{H}_i(\cdot)}{\partial q_i(t)} = \frac{a \sum_{j \neq} q_j(t)}{\left[q_i(t) + \sum_{j \neq} q_j(t) \right]^2} - c - \lambda_{ii}(t) = 0; \tag{4.2}$$

$$-\frac{\partial \mathcal{H}_i(\cdot)}{\partial k_i(t)} = \dot{\lambda}_{ii}(t) - \rho \lambda_{ii}(t) \Leftrightarrow$$

[7] Exponential discounting is omitted for the sake of brevity.

$$\dot{\lambda}_{ii}(t) = b - \lambda_{ii}(t)(A - \rho - \delta) \tag{4.3}$$

with the transversality condition

$$\lim_{t\to\infty} e^{-\rho t}\lambda_{ii}(t)\,k_i(t) = 0. \tag{4.4}$$

At this point, it is worth noting that the $N-1$ co-state equations pertaining to any $\lambda_{ij}(t)$, with $j \neq i$, are omitted as they are irrelevant due to the fact that the game exhibits separate state equations, i.e., the state dynamics of any firm is independent of the rivals' states and controls. Hence, any co-state equation

$$-\frac{\partial \mathcal{H}_i(\cdot)}{\partial k_j(t)} = \dot{\lambda}_{ij}(t) - \rho\lambda_{ij}(t) \tag{4.5}$$

indeed admits the solution $\lambda_{ij}(t) = 0$ at all times, for all $j \neq i$.

This, together with the fact that the Hamiltonian function (4.1) of the generic firm i is linear in the vector of states $\mathbf{k}(t)$, immediately implies the following result:[8]

Proposition 4.1. *The differential game is a linear state one. Therefore, the open-loop equilibrium is subgame perfect as it coincides with the feedback equilibrium yielded by the Bellman equation.*

Proof. See Appendix 1.■

Before proceeding, it is worth noting that comparing (4.2) and (2.3), one immediately sees that the presence of capital accumulation in the dynamic game plays a key role in opening the way towards a solution to the indeterminacy issue affecting the static game as marginal cost c tends to zero,

[8] Proposition 4.1 would hold true also in the more general case where $y_i(t) = f(k_i(t))$, with $f'(k_i(t)) > 0$ and $f''(k_i(t)) \leq 0$. That is, state-linearity is not necessary to yield subgame perfection in a Cournot-Ramsey game. I am using the $A-k$ version just to simplify the exposition. For more on this issue, see Cellini and Lambertini (1998, 2008).

precisely because of the fact that the co-state variable that firm i attaches to its own capacity accumulation dynamics enters the FOC on the investment control. I.e., (4.2) admits a viable solution even if $c = 0$, provided $\lambda_{ii}(t)$ is non-nil. As we will see in the reminder, this is precisely the case.

From (4.2), one obtains the expression of the co-state variable λ_{ii}:[9]

$$\lambda_{ii} = \frac{a \sum_{j \neq i} q_j}{\left(q_i + \sum_{j \neq} q_j\right)^2} - c. \tag{4.6}$$

Then, differentiating the above expression w.r.t. time yields:

$$\dot{\lambda}_{ii} = \frac{a \sum_{j \neq i} \dot{q}_j}{\left(q_i + \sum_{j \neq i} q_j\right)^2} - \frac{2a \sum_{j \neq i} q_j \left(\dot{q}_i + \sum_{j \neq i} \dot{q}_j\right)}{\left(q_i + \sum_{j \neq i} q_j\right)^3} \tag{4.7}$$

which, using (4.6) and imposing symmetry across control, state and co-state variables, yields the following control dynamics:

$$\dot{q} = \frac{q\left[a(N-1)(A-\rho-\delta) + N^2 q\left(c(A-\rho-\delta) - b\right)\right]}{a(N-1)}. \tag{4.8}$$

Now observe that, if $b = 0$, the above equation becomes

$$\dot{q} = \frac{q\left[a(N-1) - cN^2 q\right](A-\rho-\delta)}{a(N-1)}, \tag{4.9}$$

with the stationarity condition $\dot{q} = 0$ being satisfied by

$$q = 0\,;\ \widetilde{q} = \frac{a(N-1)}{cN^2}\,;\ A = \rho + \delta, \tag{4.10}$$

where (i) $q = 0$ implies that firms don't sell, and therefore their equilibrium profits are obviously nil; the second solution, $\widetilde{q}$, coincides with that of the

[9] In the remainder of the paper, I will omit the explicit indication of the time argument for brevity.

static game illustrated in section 2, and therefore is an acceptable solution only if marginal cost c is strictly positive; and $f'(k) = \rho + \delta$ is the *Ramsey golden rule*.

Instead, for all $b > 0$, The Ramsey solution disappears and imposing the stationarity condition on (4.8) yields:

$$q = 0;\ \widehat{q} = \frac{a(N-1)(A-\rho-\delta)}{N^2[b+c(A-\rho-\delta)]} \tag{4.11}$$

with the second solution being admissible even if c were nil. The above solution immediately proves the following:

Remark 4.1. *The steady state output of the differential game is admissible for all c, including $c = 0$.*

The output $\widehat{q}$ can be plugged into (3.2) to impose stationarity on the capital accumulation process. This yields:

$$k = \frac{a(N-1)(A-\rho-\delta)}{N^2[b+c(A-\rho-\delta)](A-\delta)}. \tag{4.12}$$

Moreover,

$$\widetilde{q} - \widehat{q} = \frac{a(N-1)b}{N^2[b+c(A-\rho-\delta)]c} > 0\,\forall b > 0, \tag{4.13}$$

showing that the static Cournot-Nash output is strictly larger than the open-loop (or feedback) equilibrium output for all positive levels of the opportunity cost b.

Using $\widehat{q}$, steady state individual profits simplify as follows:

$$\pi^{ss} = \frac{a[(A-\delta)(A-\rho-\delta)+\rho(b(N-1)-c(A-\delta))]}{N^2[b+c(A-\rho-\delta)](A-\delta)}, \tag{4.14}$$

with

$$\pi^{ss} - \pi^{CN} = \frac{ab(N-1)\rho}{N^2[b+c(A-\rho-\delta)](A-\delta)} > 0 \tag{4.15}$$

for all positive b, ρ. Hence, I may state:

Proposition 4.2. *At the subgame perfect equilibrium of the dynamic game, with*

$$k^{ss} = \frac{a(N-1)(A-\rho-\delta)}{N^2[b+c(A-\rho-\delta)](A-\delta)}\,;\; q^{ss} = \frac{a(N-1)(A-\rho-\delta)}{N^2[b+c(A-\rho-\delta)]}$$

the representative firm produces less and earns higher profits than at the Cournot-Nash equilibrium of the static game, for all positive levels of the discount rate and opportunity cost.

There remain to assess the stability properties of the steady state equilibrium:

Proposition 4.3. *The steady state solution*

$$k^{ss} = \frac{a(N-1)(A-\rho-\delta)}{N^2[b+c(A-\rho-\delta)](A-\delta)}\,;\; q^{ss} = \frac{a(N-1)(A-\rho-\delta)}{N^2[b+c(A-\rho-\delta)]}$$

is a saddle point equilibrium for all $A > \delta + \rho$.

Proof. See Appendix 2.■

Having characterised the subgame perfect equilibrium of the differential game, I may now proceed to the analysis of its application to horizontal mergers.

5 Application: horizontal mergers

To illustrate the advantages of our approach to the feedback solution of the differential oligopoly game *à la* Ramsey, we illustrate here its applicability to the analysis of the private profitability of a horizontal merger, and its welfare appraisal.

As is well known, a lively debate has taken place on this topic from the 1980's, based upon static oligopoly models. A thorough overview of it is outside the scope of the present paper, and it will suffice to recollect a few essential aspects. Examining a Cournot industry with constant returns to scale, Salant *et al.* (1983) have shown that a large proportion of the population of firms has to participate in the merger in order for the latter to be profitable. In particular, a striking result of their analysis is that, in the triopoly case, bilateral mergers are never profitable. Enriching the picture by allowing for the presence of convex variable costs and fixed costs, one may find a way out of this puzzle (see Perry and Porter, 1985; and Farrell and Shapiro, 1990).

Now take the static Cournot game and examine the incentive for M firms to merge horizontally, out of the initial N. After the merger (if it does take place), there remain $N-M+1$ firms. The merger is profitable iff

$$\frac{\pi^{CN}(N-M+1)}{M}=\frac{a^2}{M(N-M+1)^2}>\pi^{CN}(N)=\frac{a}{N^2} \tag{5.1}$$

that is, iff

$$N^2-M^2+M(2N+1)>0 \tag{5.2}$$

which holds for all

$$M\in\left(\frac{1+2N-\sqrt{4N+1}}{2},\frac{1+2N+\sqrt{4N+1}}{2}\right). \tag{5.3}$$

It is easily checked that, if $N=3$ and $M=2$, the merger is profitable.

If instead we consider the steady state outcome of the differential game, the profit incentive for an M-firm merger is measured by

$$\frac{\pi^{ss}(N-M+1)}{M}>\pi^{ss}(N). \tag{5.4}$$

The above condition is satisfied for all

$$M\in\left(\frac{1}{2}+N-\sqrt{\Psi},\frac{1}{2}+N+\sqrt{\Psi}\right) \tag{5.5}$$

where

$$\Psi \equiv [b + c(A-\delta)(A-\rho-\delta) + Nb\rho]\,[c\delta(\rho+\delta)(4N+1) + \tag{5.6}$$

$$A((b - c(2\delta+\rho))(4N+1) + Ac)(4N+1) - b(\delta+\rho+N(4\delta+3\rho))]\,.$$

Next, one can compare the interval (5.5) against (5.3), to verify the following properties:

$$\frac{\partial\left(1/2 + N + \sqrt{\Psi}\right)}{\partial b} \quad \propto \quad -cN^2(A-\delta)(A-\rho-\delta)\rho < 0 \tag{5.7}$$

$$\frac{\partial\left(1/2 + N - \sqrt{\Psi}\right)}{\partial b} \quad \propto \quad cN^2(A-\delta)(A-\rho-\delta)\rho > 0 \tag{5.8}$$

$$\lim_{b\to 0}\frac{1}{2} + N + \sqrt{\Psi} \quad = \quad \frac{1+2N+\sqrt{4N+1}}{2} \tag{5.9}$$

$$\lim_{b\to 0}\frac{1}{2} + N - \sqrt{\Psi} \quad = \quad \frac{1+2N-\sqrt{4N+1}}{2} \tag{5.10}$$

Taken together, these facts entail that the interval wherein the M-firm merger is profitable is wider in the dynamic setup than in the static one. Only in the limit, where the opportunity cost of holding installed capacity drops to zero, these two intervals do coincide.[10] This ultimately entails that taking properly into account (i) the possibility that firms accumulate capacity, and (ii) the related aspect that this is in general a costly activity, reveals that horizontal mergers appearing unfeasible in the static game become feasible in the dynamic one. The examination of the welfare consequences of a merger is omitted, as it goes without saying that any merger would diminish social welfare, both in the static as well as in the dynamic setting. This is

[10]Conversely, the same result applies, for all $b > 0$, by setting either $\rho = 0$ or taking the limit for ρ growing up to infinity.

trivially due to the fact that the damage caused to consumer surplus always outweighs the increase in industry profits.[11]

6 Concluding remarks

I have characterised the subgame perfect equilibrium of a dynamic Cournot game with hyperbolic demand and costly capacity accumulation, showing that the open-loop solution is subgame perfect. Then, I have employed the model to analyse the feasibility of horizontal mergers, and compare the result stemming from the steady state of the differential game against the merger incentive associated with the static version of the model. There emerges that allowing for the role of time in determining firms' incentives as to their optimal long-run size makes mergers, in general, more likely to take place (and therefore more dangerous) than they would be if judging on the basis of the static approach.

[11]In line of principle, a merger could allow for some reduction in the total opportunity costs for the industry, giving rise to a possible *efficiency defense* argument (see Farrell and Shapiro, 1990). Although I omit the related calculations for brevity, it is quickly checked that this never outweighs the loss in consumer suplus necessarily generated by any merger. Hence, in this model the efficiency argument cannot be advocated to justify the merger itself.

Appendices

Appendix 1. Proof of Proposition 4.1

The observation that the differential game under consideration is indeed a linear state one suffices to prove the claim.[12] However, it is interesting to show that the game is indeed solvable using the corresponding Bellman equation:[13]

$$\rho V_i(\mathbf{k}) = \max_{q_i} \left(\pi_i + \frac{\partial V_i(\mathbf{k})}{\partial k_i} + \sum_{j \neq i} \frac{\partial V_i(\mathbf{k})}{\partial k_j} \right) \tag{a1}$$

with a linear value function, notwithstanding the presence of a non-linear demand function.

To prove this result, take

$$V_i(\mathbf{k}) = e_i k_i + \sum_{j \neq i} e_j k_j + \theta, \tag{a2}$$

so that $\partial V_i(\mathbf{k}) / \partial k_i = e_i$ for all i. The first order condition taken on (a1) is:

$$\frac{a \sum_{j \neq} q_j(t)}{\left[q_i(t) + \sum_{j \neq} q_j(t) \right]^2} - c - e_i = 0 \tag{a3}$$

i.e., the same as (4.2) except for the appearance of e_i in place of the co-state λ_{ii} (I'll come back to this aspect in the remainder of the proof).

[12] A linear state game is one where (using the same symbols as in this paper, to indicate states and controls):

$$\frac{\partial^2 \mathcal{H}_i(\cdot)}{\partial q_i(t)\, \partial k_j(t)} = \frac{\partial^2 \mathcal{H}_i(\cdot)}{\partial k_j^2(t)} = 0$$

for all i, j. For more on linear state games, see Dockner *et al.* (2000, ch. 7), *inter alia*.

[13] Throughout the Appendix, I will omit the explicit indication of the time argument for brevity.

To proceed with the analytical solution of the feedback problem, I introduce two symmetry conditions: one is $q_i = q_j$ for all j, while the other one is $k_j = \overline{k}$ (and also $e_j = \overline{e}$) for all $j \neq i$. The former says that the equilibrium output must be symmetric across all firms, the second states that, from the standpoint of a generic firm i, the rivals' capacities (and therefore also their weights in the value function) must be symmetric. In introducing the second condition I explicitly refrain from setting $k_i = k_j = \overline{k}$ and $e_i = e_j = \overline{e}$ as the relative weight of firm i's capacity is in its own right different from the rivals'. By doing so I would unduly introduce some degree of collusion in a game that is strictly noncooperative.

Using the symmetry condition on quantities, (a3) yields:

$$q_i^* = \frac{a\,(N-1)}{N^2\,(c+e_i)} \tag{a4}$$

so that (a2) can be rewritten as follows:

$$\begin{aligned} & k_i\left[b - e_i\,(A-\delta-\rho)\right] - \overline{k}\left[(A-\delta)\,(N-1) - 2\rho\right]\overline{e} + \\ & + \frac{N^2\,(c+e_i)\,\theta\rho - a\left[c+e_i-\overline{e}\,(N+1)^2\right]}{N^2\,(c+e_i)} = 0 \end{aligned} \tag{a5}$$

giving rise to a system of three equations:

$$\begin{aligned} N^2\,(c+e_i)\,\theta\rho - a\left[c+e_i-\overline{e}\,(N+1)^2\right] &= 0 \\ \left[(A-\delta)\,(N-1)-2\rho\right]\overline{e} &= 0 \\ \left[b - e_i\,(A-\delta-\rho)\right] &= 0 \end{aligned} \tag{a6}$$

to be solved w.r.t. the coefficients of the Bellman equation, e_i, $\overline{e}$ and θ. This yields:

$$\theta = \frac{a\left[c+e_i-\overline{e}\,(N+1)^2\right]}{N^2\,(c+e_i)\,\rho};\ \overline{e} = 0;\ e_i = \frac{b}{A-\delta-\rho}. \tag{a7}$$

Hence, the resulting feedback equilibrium output is

$$q_F = \frac{a(N-1)(A-\rho-\delta)}{N^2[b+c(A-\rho-\delta)]} = q^{ss} \tag{a8}$$

and obviously the optimal capacity endowment at the feedback equilibrium coincides with k^{ss}.

This concludes the proof of Proposition 4.1. As an ancillary observation, it is worth noting that here, precisely because the open-loop solution is indeed a degenerate feedback one, the co-state variable λ_{ii} appearing in the open-loop formulation of the game can be appropriately considered as a shadow price (of an additional unit of capacity, in the present setup), while, in general, this is true only of the partial derivative of the value function, $\partial V_i(\mathbf{k})/\partial k_i$ (for more on this aspect, see Caputo, 2007).■

Appendix 2. Proof of Proposition 4.3

The stability properties of the dynamic state-control system must be evaluated by assessing the trace and determinant of the 2×2 Jacobian matrix:

$$J = \begin{bmatrix} \frac{\partial \dot{k}}{\partial k} & \frac{\partial \dot{k}}{\partial q} \\ \frac{\partial \dot{q}}{\partial k} & \frac{\partial \dot{q}}{\partial q} \end{bmatrix} \tag{a9}$$

that, in correspondence of the symmetric steady state equilibrium, exhibits the following determinant:

$$\Delta(J) = -(A-\delta)(A-\delta-\rho). \tag{a10}$$

The above expression is negative for all $A > \delta + \rho$. Accordingly, in such a range the steady state (k^{ss}, q^{ss}) is a saddle point equilibrium.■

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