The emergence of norms of cooperation in stag hunt games with production

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Abstract

In this paper we study a two agents asymmetric stag hunt game. The model has an infinity of strict, Pareto rankable Nash equilibria. The equilibrium selection problem is solved by appealing to the stochastic stability concept put forward by Young (1993). We prove two main results. When the action sets are numerable infinite sets, then for any value of the distributive parameter we can expect the emergence of a norm involving less than maximal cooperation. When instead the action sets are finite sets of a particular type (in the sense that each agent can choose his maximum optimal effort and fractions of this), then for some value of the distributive parameter we can expect the emergence of a norm involving maximal cooperation.

keywords: asymmetric stag hunt game; stochastic stability; cooperation norms.

1 Introduction

According to Skyrms (2004), the stag hunt is a story that became a game. The game is a prototype of the social contract while the story is told by Rousseau. Consider two hunters who have to decide whether to cooperate to hunt a stag. Suppose that, in the case they succeed in hunting the stag, the catch is divided equally. Hunting stags is demanding and it requires the cooperation of both. Suppose that, while waiting for the stag, a hare happens to pass within reach of one of them; hunting hares is much easier: it requires a minimum effort and it can be done successfully without the cooperation of the other hunter. Let us assume that no binding agreement is possible for the two hunters. Although for each of them half a stag is more valuable than a hare, they can not be sure that the other player will provide the required effort. In other terms, although the situation in which both hunt the stag is a Nash equilibrium which Pareto dominates the other equilibrium in which both hunt the hare (the minimax

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solution), the former equilibrium may fail to be risk dominant. See Carlsson and Van Damme (1993).

In this paper we follow Bryant (1983) and Cooper (1999) and use the stag hunt game as a model of team production. We assume that agents' effort are complementary inputs so that total output of the team is determined by the least effort. In terms of the stag hunt parable, this means that we focus on the total catch of the hunters rather than on the stag/hare alternative. It is a standard result in games with strategic complementarities that, when the marginal benefit from coordinated actions is greater than the marginal cost of effort, then any common level of effort is a strict Nash equilibrium; see Cooper and John (1988). Moreover, since all individuals prefer an equilibrium in which all players supply higher effort, all these Nash equilibria can be Pareto ranked. We are thus in a situation in which since the Nash equilibrium concept neither prescribes nor predicts the outcome of the game, it needs to be supplemented with an adequate theory of equilibrium selection. Any traditional refinement would not help as long as the play is simultaneous; of course if we consider a sequential game, then the only subgame perfect equilibrium is the Pareto optimum one. However, the evidence from experimental economics suggests that the Pareto dominant equilibrium is quite unlikely in the simultaneous stag hunt game; in this case, in fact, life can indeed be "inside the production possibility frontier" (Cooper (1999), pg. 151). See Van Huyck, Battalio and Beil (1990), Van Huyck, Cook and Battalio (1997) and the discussion in Crawford (1991, 1995).

We depart from Bryant (1983) in three respects. First we consider a two agents economy with identical (separable) utility functions but different productivities. Second, although the distributive parameter (x) regulating the distribution of the joint product among the two agents is fixed[1], we do not assume from the outset that the resulting distribution is egalitarian.[2] Third, we assume that our stag hunt game is played by boundedly rational, randomly matched players, along the lines suggested by Young (1993, 1998). In a sense we are considering a stag hunt played by boundedly rational "strangers".[3]

Since our strangers are engaged in a strategic game, they need to form an expectation on the behavior of their opponent. Following Young we consider the case in which this expectation is shaped by the accumulation of antecedents, according to an inductive process. Suppose that each agent collects a sample of size k from the last m past plays of the game, with $k < m$. Given k he then extracts the empirical frequencies with which each (pure) strategy was played in the past by other agents. With probability $1 - \epsilon$ each agent then chooses an

[1] We assume that it is determined by the existing distributional rule (left unexplained) endorsed by the society.

[2] As pointed out by Cooper (1999), the coordination problems arising in Byant's (1983) model is partly a consequence of the rule that distributes equally the fruits of the cooperation regardless of individual effort levels. See also Bryant (1994).

[3] This last assumption makes our contribution departing also from Crawford's (1991) evolutionary approach to the stag hunt as well as from Crawford's (1995).

action which is a best reply to these empirical frequencies while with probability ϵ each agent makes a mistake, i.e. he chooses an action which is not a best reply. The strategies chosen in the current period are recorded and in the next period the game will be played, along the same lines, by another draw of agents from the same population. Following Young (1993) we say that a state is stochastically stable if, in the long run, it can be observed with positive probability when the probability of mistakes is small and the sample is sufficiently limited. When there is a unique stochastically stable state this is the equilibrium that, in the long run, will be observed with probability close to one so that it becomes the conventional way of playing the game. In this sense the approach gives us a theory of equilibrium selection.

We first analyze the case in which agents can choose among a *continuum* of pure strategies (i.e. effort levels $e_i \in [0, e_i^{\max}]$) Let $\widehat{e}(x)$ denote the Nash equilibrium where $\widehat{e}_i(x) \in [0, \widehat{e}_i^{\max}(x)]$. For any x we have a continuum of Nash equilibria which can be Pareto ranked. We say that a Nash equilibrium involves *maximal cooperation* if at least one agent supplies his maximum optimal effort, that is $\widehat{e}_i(x) = \widehat{e}_i^{\max}(x)$.

Our first result says that, for any value of the distributive parameter, interactions of boundedly rational strangers converge to a stochastically stable state $\left(e_1^S(x), e_2^S(x)\right)$ which, since it involves less that maximal cooperation (i.e. $e_i^S(x) < \widehat{e}_i^{\max}(x)$) is not Pareto efficient; the precise state to which our economy converges depends on the value of the distributive parameter. However, we also show that at any stochastically stable state each agent supplies an effort which is never smaller than half of his optimal maximum level. Hence, since $\frac{1}{2}\widehat{e}_i^{\max}(x) \leq e_i^S(x) < \widehat{e}_i^{\max}(x)$, we say that in our economy the stochastically stable state involves a *minimal cooperation*. This is the equilibrium that is easiest to flow into from all other states in the sense that it is more robust to agents' mistakes than all other equilibria; for this reason it tends to persists and becomes a conventional way of playing the stag hunt game for our strangers (or a conventional social contract). This is quite surprising because, since our strangers can choose among a continuum of actions, they are in the case in which the risk of miscoordination is the highest. Nevertheless our first result, far from suggesting that the social contract "might degenerate spontaneously into the state of nature" as claimed by Skyrms (2004; pg. 12), tells us that we can quite confidently expect the emergence of a norm involving minimal (but positive) cooperation. To return to the stag hunt metaphor, it is true that at this stochastically stable state the hunters do not get the maximal catch they could; however, each gets more than the catch he could get alone (i.e. in the state of nature).

Our first result also suggests that it can effectively be difficult for our agents to improve upon the conventional social contract. In fact, although our strangers may realize (peraphs with the help of an external observer) that, for the given value of the distributive parameter, a larger pie can be achieved if they supply higher optimal effort (thus resulting in an equilibrium which Pareto dominates the stochastically stable one), the resulting equilibrium is not stochastically stable.[4]

We then analyze the interactions of boundedly rational strangers when each can choose among a *finite* number of pure strategies. Our second result says that, although equilibria with minimal cooperation may still be stochastically stable, now it is also possible to observe the emergence of a norm involving maximal cooperation. However the emergence of this norm is not due to any efficiency considerations; in fact, we show that the stochastically stable equilibrium involving maximal cooperation is Pareto efficient for some values of the distributive parameter only and provided that the number of strategies available is not too big and agents have sufficiently different productivities. Lastly we show that when each agent can choose more that two actions, the equilibrium with maximal cooperation corresponding to the value of the distributive parameter suggested by the utilitarian unweighted cooperative solution will never be stochastically stable. We remark that our results are not due to any incentive problem; see Legros and Mathews (1993), Vislie (1994) and Hvide (2001).

The remaining of the paper is organized as follow. In Section 2 we present a variant of Bryant (1983) symmetric coordination problem. In Section 3 we introduce our asymmetric coordination game. In Section 4 we briefly summarize Young's (1993) concept of stochastic stabile state. Section 5 then discusses the stochastic stability of our asymmetric coordination game first when agents have a continuum of strategies and then when agents can choose only among a finite number of discrete strategies. Section 7 summarizes our results.

2 A symmetric coordination game

In this section we briefly present a variant of Bryant's (1983) coordination game. Consider two equally productive agents engaged in a joint project and let

$$Y = \alpha \min[e_1, e_2]$$

be the technology available, where $e_i \in [1, e^{\max}]$. Suppose that the outcome of the cooperation is divided according to the distributional parameter x so that agent 1 gets $Y_1 = xY$ and agent 2 gets $Y_2 = (1-x)Y$. Let us first suppose[5] $x = 1/2$. Denoting by $V_i(Y_i, e_i) = Y_i - be_i$ the payoff of the generic agent i, we get

$$V_i = \tfrac{\alpha}{2} \min[e_1, e_2] - be_i = a \min[e_1, e_2] - be_i,$$

[4] One possibility open to our strangers would be to agree on conditional contracts, whose enforcement is ensured by an external observer, in which (a) they agree on a particular division of the fruits of social cooperation and (b) they supply their maximal optimal effort, given this value of the distributive parameter. However this alternative is not viable in our economy since there is no external observer. A still different alternative would be to consider self-enforcing social contracts, as in Binmore (1994). We leave the exploration of this alternative to future research.

[5] As it can be verified, $x = 1/2$ is also to the cooperative solution of the game. See Cooper (1999).

where we assume $a > b$. This is the game studied by Van Huyck et al. (1990). Because of the technological complementarity, and since effort is costly, no agent has the incentive to choose an effort such that his contribution to the joint project is larger than the contribution of the other agent. Let $e_2 = \widetilde{e}$. The payoff of agent 1 is $V_1(\widetilde{e}, \widetilde{e}) = (a-b)\,\widetilde{e}$ if $e_1 = \widetilde{e}$ and $V_1(e_1, \widetilde{e}) = (a-b)\,e_1$ if $e_1 < \widetilde{e}$. Analogously for agent 2. Since $V_1(\widetilde{e}, \widetilde{e}) - V_1(e_1, \widetilde{e}) = (\widetilde{e} - e_1)(a-b)$, it follows that all the profiles $(e_1, e_2) = (e, e)$ with $e \leq e^{\max}$ are Nash equilibria, being $a > b$.

Suppose now that the distributive parameter is x. Then

$$
\begin{aligned}
V_1 &= \alpha x \min[e_1, e_2] - b e_1 \\
V_2 &= \alpha(1-x) \min[e_1, e_2] - b e_2.
\end{aligned}
$$

Let $e_2 = \widetilde{e}$. The payoff of agent 1 is $V_1(\widetilde{e}, \widetilde{e}) = (\alpha x - b)\,\widetilde{e}$ if $e_1 = \widetilde{e}$ and $V_1(e_1, \widetilde{e}) = (\alpha x - b)\,e_1$ if $e_1 < \widetilde{e}$. Hence $V_1(\widetilde{e}, \widetilde{e}) \geq V_1(e_1, \widetilde{e})$ if $(\widetilde{e} - e_1)(\alpha x - b) \geq 0$, that is if $x \geq b/\alpha$. Consider now player 2 and let $e_1 = \widetilde{e}$. Then $V_2(\widetilde{e}, \widetilde{e}) = (\alpha(1-x) - b)\,\widetilde{e}$ if $e_2 = \widetilde{e}$ and $V_2(e_2, \widetilde{e}) = (\alpha(1-x) - b)\,e_2$ if $e_2 < \widetilde{e}$. Hence $V_2(\widetilde{e}, \widetilde{e}) \geq V_2(e_2, \widetilde{e})$ if $(\widetilde{e} - e_2)(\alpha(1-x) - b) \geq 0$, that is if $x \geq (\alpha - b)/\alpha$. It turns out that all the profiles $(e_1, e_2) = (e, e)$ with $e \leq e^{\max}$ are Nash equilibria if and only if $b/\alpha \leq x \leq (\alpha - b)/\alpha$. In this case the game is a stag hunt. However if the distributive parameter does not satisfies this condition, then the game admits only one Nash equilibrium in which $(e_1, e_2) = (0, 0)$.

3 An asymmetric coordination game

In this section we modify the basic model by allowing some form of heterogeneity; specifically we assume (a) that individual efforts are not equally productive and (b) that the distribution of the fruits of joint production is not a priori egalitarian. Moreover, since we want to consider a game that maintains the structure of the stag hunt for any value of the distributive parameter, we modify the Van Huyck et al. (1990) model by considering a non linear effort disutility. In the next section we shall use this model to study the problem of equilibrium selection by appealing to the inductive argument put forward by Young (1993).

As in Cooper (1999), we consider an economy populated by two individuals engaged in a joint project. The output produced is given by

$$
Y = \min[\alpha e_1, \beta e_2] \tag{1}
$$

where e_1 and e_2 denote the effort levels chosen by the two agents while α and β are two real numbers representing the (possibly) heterogeneous individuals' productivities. We suppose that there is an already established distributional rule which determines how the output is shared by the two individuals; letting x denote the share of the production going to agent 1, we get $Y_1 = xY$ and $Y_2 = (1-x)\,Y$. The output received by each individual is entirely consumed.

Let[6]

$$\begin{aligned} V_1 &= x\min[\alpha e_1, \beta e_2] - e_1^2 \\ V_2 &= (1-x)\min[\alpha e_1, \beta e_2] - e_2^2. \end{aligned} \tag{2}$$

denote the agents' payoffs and let $e_1^{\max} = \alpha/2$ and $e_2^{\max} = \beta/2$ be the maximum feasible level of efforts for the two agents. Let G denote the game in which agents simultaneously selects their effort levels from the sets $S_i = [0, e_i^{\max}]$ and receive a payoff given by (2). Given the distributional parameter, and given agent's 2 effort, the problem faced by agent 1 is to choose[7] e_1 to maximize $x\alpha e_1 - e_1^2$ subjected to $\alpha e_1 \leq \beta e_2$. Let $\widehat{e}_1$ be the solution of this problem, where:

$$\widehat{e}_1 = \begin{cases} x\frac{\alpha}{2} & if \quad e_2 \geq x\frac{\alpha^2}{2\beta}, \\ \frac{\beta}{\alpha}e_2 & if \quad e_2 \leq x\frac{\alpha^2}{2\beta}. \end{cases} \tag{3}$$

Analogously, the problem faced by agent 2 is to choose e_2 to maximize $(1-x)\beta e_2 - e_2^2$ subjected to $\beta e_2 \leq \alpha e_1$. Let $\widehat{e}_2$ be the solution of this problem, where:

$$\widehat{e}_2 = \begin{cases} (1-x)\frac{\beta}{2} & if \quad e_1 \geq (1-x)\frac{\beta^2}{2\alpha}, \\ \frac{\alpha}{\beta}e_1 & if \quad e_1 \leq (1-x)\frac{\beta^2}{2\alpha}. \end{cases} \tag{4}$$

We can state the following result.[8]

Proposition 1 *For any $x \in [0,1]$ there is an infinity of strict, pure strategies Pareto rankable Nash equilibria. Let $\theta \equiv \frac{\alpha}{\beta}$. Then:*

(a) the set of Nash equilibria is $(\widehat{e}_1, \theta\widehat{e}_1)$ where $\widehat{e}_1 \leq \min\left(\widehat{e}_1^{\max}, \theta^{-1}\widehat{e}_2^{\max}\right) = \min\left(\frac{\alpha x}{2}, \frac{(1-x)\beta^2}{2\alpha}\right)$;

(b) for any given x, the equilibrium in which at least one agent offers his maximum optimal effort, i.e. $\widehat{e}_1 = \widehat{e}_1^{\max} = xe_1^{\max}$ and $\widehat{e}_2 = \widehat{e}_2^{\max} = (1-x)e_2^{\max}$, is Pareto dominant.

Proof. See the Appendix.

For each player we can write the payoff corresponding to any Nash equilibrium as:

$$\begin{aligned} V_1(x) &= \widehat{e}_1(x)(x\alpha - \widehat{e}_1(x)) \\ V_2(x) &= \widehat{e}_1(x)\left((1-x)\alpha - \theta^2\widehat{e}_1(x)\right) \end{aligned} \tag{5}$$

[6]The quasi-linearity with respect to the consumption good is essential because in our economy the consumption good is the numeraire; cfr. Ray et al.(2006).

[7]We restrict our analysis to the case of pure strategies only.

[8]In a similar model, Anderson et al. (2001) argue that a change in effort cost does not affect the Nash equilibria. This is not necessarily true in our model.

In this section we restrict our analysis to the case in which, for any value of the distributive parameter, at least one agent chooses his maximum level of optimal effort[9], i.e. $\widehat{e}_i = \widehat{e}_i^{\max}$. We say that the corresponding Nash equilibrium involves maximal cooperation. From (1) and Proposition 1 it follows that at any Nash equilibrium the level of production with maximal cooperation is

$$Y = \begin{cases} \alpha\widehat{e}_1^{\max} = x\frac{\alpha^2}{2} & if \quad x \leq x^*; \\ \\ \beta\widehat{e}_2^{\max} = (1-x)\frac{\beta^2}{2} & if \quad x \geq x^*, \end{cases} \tag{6}$$

where $x^* \equiv \frac{\beta^2}{\alpha^2+\beta^2} = \frac{1}{\theta^2+1}$. Production attains its maximum level $\overline{Y}$ when $x = x^*$. The following table shows the income and the utility distributions as functions of x, corresponding to the Nash equilibrium with maximal cooperation.

	$x \leq x^*$	$x \geq x^*$
$Y_1(x)$	$x^2\frac{\alpha^2}{2}$	$x(1-x)\frac{\beta^2}{2}$
$Y_2(x)$	$x(1-x)\frac{\alpha^2}{2}$	$(1-x)^2\frac{\beta^2}{2}$
$V_1(x)$	$x^2\frac{\alpha^2}{4}$	$(1-x)\frac{\beta^2}{2}\left[x\left(1+\frac{\beta^2}{2\alpha^2}\right)-\frac{\beta^2}{2\alpha^2}\right]$
$V_2(x)$	$x\frac{\alpha^2}{2}\left(1-x\left(\frac{\alpha^2}{2\beta^2}+1\right)\right)$	$(1-x)^2\frac{\beta^2}{4}$

(7)

The maximum optimal effort is supplied by agent 1 when $x \leq x^*$ and by agent 2 when $x \geq x^*$. Notice that x^* corresponds to the unweighted utilitarian cooperative solution of the model. The next Lemma establishes some relevant properties of the equilibrium payoffs functions $V_1(x)$ and $V_2(x)$.

Lemma 2 *Consider the payoff functions $V_1(x)$ and $V_2(x)$ given by (5). Then:*

a) $V_1(x) = V_2(x) = 0$ for $x = 0$ and $x = 1$. For any $0 < x \leq x^*$, $V_1(x)$ is an increasing and convex function while for any $x \geq x^*$, $V_1(x)$ is a concave function with a maximum $\overline{V}_1$ at $x = x_1$. For any $0 < x \leq x^*$, $V_2(x)$ is a concave function with a maximum $\overline{V}_2$ at $x = x_2$ while for any $x \geq x^*$, $V_2(x)$ is a decreasing and convex function. $V_1(x)$ and $V_2(x)$ are maximized respectively when

$$x_1 = \frac{\beta^2+\alpha^2}{2\alpha^2+\beta^2}, \qquad x_2 = \frac{\beta^2}{\alpha^2+2\beta^2}, \tag{8}$$

where, for any (α,β), $x_2 < x^* < x_1$ and $x_2 < \frac{1}{2} < x_1$.

[9] Notice that since Bryant (1983) considers only an egalitarian distribution, he finds that there is only one Pareto dominant equilibrium. In our case instead, we have one Pareto dominant equilibrium for any x.

b) Let $\alpha > \beta$. Then $\overline{V}_1 < \overline{V}_2$. Moreover $V_2(x) > V_1(x)$ for every $x < x_3$ and $V_2(x) \leq V_1(x)$ for every $x \geq x_3$ where

$$x_3 \equiv \frac{\beta^2 + \alpha^2}{3\alpha^2 + \beta^2}, \tag{9}$$

and $x_2 < x^* < x_3$.

c) Let $\alpha < \beta$. Then $\overline{V}_1 > \overline{V}_2$. Moreover $V_2(x) > V_1(x)$ for every $x < x_4$ and $V_2(x) \leq V_1(x)$ for every $x \geq x_4$ where

$$x_4 \equiv \frac{2\beta^2}{\alpha^2 + 3\beta^2}, \tag{10}$$

and $x_4 < x^* < x_1$.

Proof. Omitted since it relies on simple algebraic manipulations. ■

The next Lemma shows that the game has two properties that will be useful in the following analysis.

Lemma 3 *The game G is acyclic and satisfies the bandwagon property. Moreover, let $L(e)$ denote the length of the shortest best reply path originated in the strategy profile e; then $L_\Gamma = \max L(e) = 2$.*

Proof. See the Appendix.

Acyclicity means that the best reply graph contains no directed cycles, a property satisfied by all coordination games. A sufficient condition for the (marginal) bandwagon property to hold for generic (i.e. not necessarily acyclic) symmetric games has been proved by Kandori and Rob (1998). A reformulation which holds for acyclic but not necessarily symmetric two players games is given by Binmore, Samuelson and Young (2003). Let $\Omega = (2, S_1, S_2, V_1, V_2)$ be a finite acyclic game with two players and let S^N denote the set of all strict Nash equilibria of the game. Ω exhibits the bandwagon property if for each $\overline{s} \in S^N$ and $s \in S$ with $\overline{s}_i \neq s_i$ for any $i = (1, 2)$ the following conditions is satisfied

$$V_1(\overline{s}_1, \overline{s}_2) - V_1(s_1, \overline{s}_2) \geq V_1(\overline{s}_1, s_2) - V_1(s_1, s_2), \tag{11}$$

with an analogous condition holding for agent 2. This essentially says that, for both agents, deviations from the equilibrium strategy are more costly when the opponent plays his part of the equilibrium.

Example 1. Consider Table (7) and let $\alpha = 2$ and $\beta = 1$. Figure 1 plots V_i as a function of the distributive parameter. The maximum value of V_1 is $1/9$ and it is achieved when $x = x_1 = 5/9$; the maximum value of V_2 is $1/6$ and it is achieved when $x = x_2 = 1/6$. Notice that $V_1 = V_2$ for $x = x_3 = 5/13$. Lastly, when $x = x^* = 0.2$ (that is when the output produced is the maximum possible), $V_1 = 0.04$ and $V_2 = 0.16$.

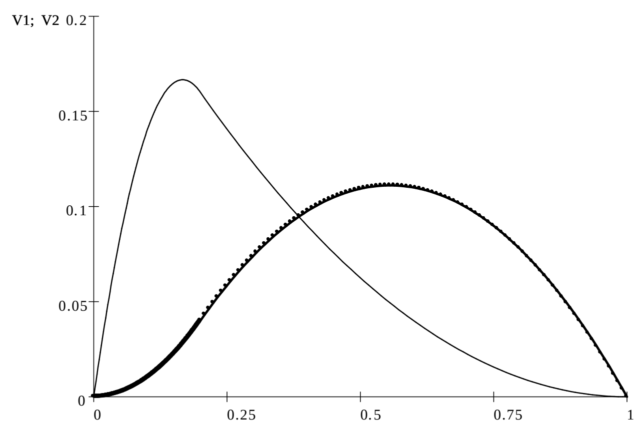

Figure 1 - V_1 and V_2 as a function of x when $\beta = 1$ and $\alpha = 2$. V_1 dotted.

Lemma 3 can be used to establish the existence of two regions in which mutually advantageous agreements are possible; these are respectively the intervals $[0, x_2)$ and $(x_1, 1]$. The interval $[x_2, x_1]$ represents instead the set of efficient distributional norms: for any x belonging to this interval we can not increase the utility of one agent without decreasing the utility of the other one.

We now ask which is the maximum utility that both agents can reach at any Nash equilibrium with maximal cooperation. In order to derive this, we can not rely on the utility possibility frontier (UPF) traditionally defined since this is based on the assumption that the amount of good available for distribution is exogenously given. This is not true in our economy. As we have seen (see the proof of Proposition 1), the utility that each agent derives at the Nash equilibrium is maximum when, for any value of the distributive parameter, at least one agent chooses his maximum optimal effort. However, since this maximum effort depends on x, by changing the distributive parameter we change the total output as well as the associated utility of each agent. This leads us to introduce the concept of utility distribution frontier (UDF) that we now define.[10]

Definition 4 *Consider the Nash equilibrium in which, for any given x, at least one agent offers his maximum optimal effort. The Utility Distribution Frontier (UDF) describes how the corresponding utility pair (V_1, V_2) varies with x.*

Since the UDF is derived under the assumption that at least one agent supplies his maximum optimal efforts, it represents the equilibria involving maximal cooperation. The UDF tells us: (a) the utility we can assure to one agent given the utility of the other one when the distributional parameter x changes, knowing that (b) to any particular x there corresponds a particular level of output

[10] See Wasow (1980).

available for distribution. As a consequence, when we move along the UDF (that is when x changes), output varies. There is however a precise relationship between the two frontiers. To see this consider a given level of output and the UPF associated. Except when the specified level of output corresponds to maximum level, the UPF intersects the UDF in two points. These correspond to the two distributions of utility (corresponding to two different values of the distributive parameter) supporting the specified level of output as an equilibrium.

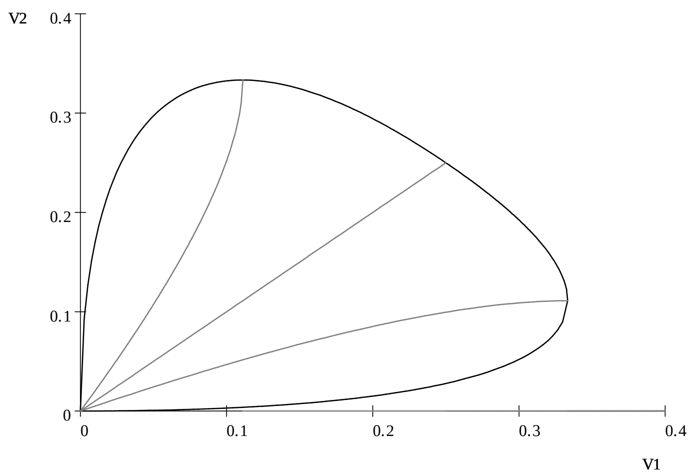

Figure 2 - UDF and equilibrium paths.

Figure 2 shows an example of the UDF (in which $\alpha = 2$ and $\beta = 1$).[11] From inspection we see immediately that the UDF is asymmetric and not comprehensive. Since we consider an economy populated by agents with the same utility functions, the asymmetry is entirely due to the difference in effort productivities. The non-comprehensiveness derives from the existence of the two regions in which mutually advantageous agreements are possible.[12]

Suppose now that x is given and consider the continuum of Nash equilibria derived by progressively increasing the effort levels, starting from the state of nature of zero effort. In the payoff space, this generates a path $V_2(V_1)$ starting from $(0, 0)$ and ending in a point on the UDF in correspondence of which at least one agent supply his maximum optimal effort. As a consequence, along the path total production increases. Figure 2 shows three possible paths, each derived for a specific value[13] of x.

[11] Let $V = (V_1, V_2)$ be a point on the UDF. Consider a clockwise change of V, starting from the origin, $x = 0$. Then output first increases; it reaches a maximum level when $x = x^*$ (which corresponds to a point V belonging to the decreasing harm of the UDF) and then declines, reaching again the origin when $x = 1$.

[12] We may be tempted to conclude that free disposal could make the UDF comprehensive; however, this is not true in our model since this would involve a series of utility pairs which are not Nash equilibria.

[13] Concerning Figure 2, the convex path on the left hand side is derived for $x = \frac{1}{3}$; the concave path on the right hand side is derived for $x = \frac{2}{3}$; lastly the linear path in the middle

is derived for $x = x^*$. In general any path with $x < x^*$ is convex in the (V_1, V_2) space whereas any path with $x > x^*$ is concave.

Lastly, we want to understand how the utility distribution frontier is affected by a change in the effort productivities. Let us keep constant the productivity of one agent and increase the productivity of the other one. Few computations show that when $x \leq x^*$ we get $\partial V_1/\partial\alpha > 0$, $\partial V_2/\partial\alpha > 0$, $\partial V_1/\partial\beta = 0$ and $\partial V_2/\partial\beta > 0$; when instead $x \geq x^*$ we get $\partial V_1/\partial\beta > 0$, $\partial V_2/\partial\beta > 0$, $\partial V_1/\partial\alpha > 0$ and $\partial V_2/\partial\alpha = 0$. This implies that the utility distribution frontier moves outward, as shown in Figure 3. Notice that for $x < x^*$, we have a multiplier effect (see Cooper and John (1988)) when the productivity of agent 1 increases; for $x > x^*$, we have instead a multiplier effect when the productivity of agent 2 increases.

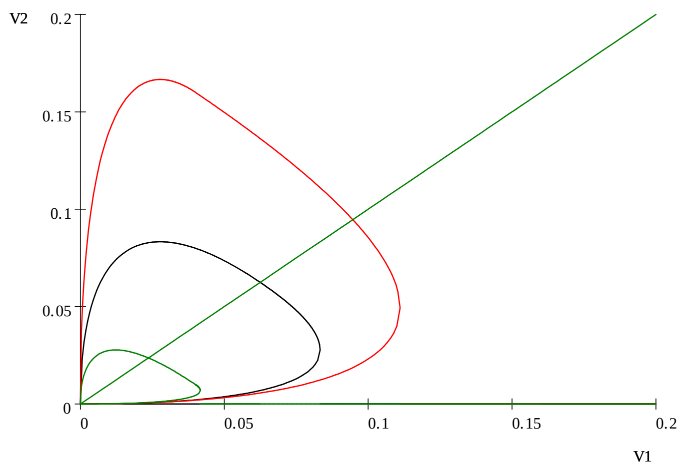

Figure 3 - Effect of a change of α on the UDF when $\beta = 1$; inner locus: $\alpha = 0.5$; middle locus: $\alpha = 1$; outer locus: $\alpha = 2$.

4 Stochastic stability

In this Section we briefly discuss the notion of stochastically stable state introduced by Young (1993).

Let $\Omega = (2, S_1, S_2, V_1, V_2)$ be a finite game with two players and let S^N denote the set of all strict Nash equilibria of the game. Let $t = 1, 2, ...$ denote successive time periods and consider a fixed (but large) population of N players. Let N_1 and N_2 be the sub-populations of agents 1 and 2 respectively. In each period the finite basic game G is played once by two agents randomly selected from these sub-populations. When selected, each agent has to form an expectation on the behavior of his opponent. Let $h(t) = (s(1), ..., s(t))$ be the history of plays at the end of period t; it consists of the past plays of the game where each play denotes the profile of strategies played in that period, i.e.$s(t) = (s_1(t), s_2(t))$. Since gathering information is costly, each agent bases his current action not on the whole past plays available but rather on a sample of k plays taken from the most recent m (with $1 \leq k < m$) plays.[14]

Thus at the beginning of period $t+1$ (with $t \geq m$) each agent consults k plays of the game and derives the empirical frequency with which each (pure) strategy was played in his sample; then he chooses his current action which is the best reply to these empirical frequencies. Current actions are recorded[15] and the economy moves from the current state h to the successor state h'. Period $t+1$ then closes.

In the new period $t+2$, the game is played again by two other randomly selected agents. As before, each consults k plays of the game from the most recent m periods and chooses his current action which is the best reply to the newly derived empirical frequencies. This moves the economy from the current state h' to the successor state, h'' and so on. The transition from one state to its successor is governed by the Markov process P^0 with transition function $P^0_{hh'}$.

Theorem 1 in Young (1993) shows that when the basic game Ω is acyclic and the sample is sufficiently limited, the adaptive process without mistakes selects a state of the form $h = (\widehat{s}, ..., \widehat{s})$ where $\widehat{s} = (\widehat{s}_1, \widehat{s}_2) \in S^N$ is a pure strategies strict Nash equilibrium profile. In this state the same Nash equilibrium profile $\widehat{s}$ is played m times in succession so that we may call it a convention.[16] Since the Markov process is reducible, the specific convention selected depends on the initial conditions.

The assumption that agents always choose a best response given the available information is clearly unrealistic. We can more realistically imagine that with probability ϵ agent i chooses an action that is not a best reply to the derived empirical frequencies (i.e. he makes a mistake) while with probability $1-\epsilon$ he chooses an action which is a best reply to the derived empirical frequencies. In this case the transition from state h to state h' is governed by the perturbed Markov process P^ϵ with transition function $P^\epsilon_{hh'}$. If we assume that all mistakes are possible and that the probability to make a mistake is time-independent, then the transition matrix associated with $P^\epsilon_{hh'}$ is strictly positive and the Markov process is irreducible and aperiodic (ergodic). The process has thus a unique stationary distribution μ^ϵ. When the probability of mistakes is small, the stationary distribution of the perturbed process P^ϵ coincides with one of the stationary distributions of the unperturbed process P^0. Hence we say that a state h is *stochastically stable* relative to the process P^ϵ if $\lim_{\epsilon \to 0} \mu^\epsilon(h) > 0$. When there is a unique stochastically stable state, this stationary distribution

[14] The fraction k/m can thus be seen as an index of the completeness of agents' information.

[15] Since the memory size is finite, the inclusion of the current period implies that agents disregards the most distant play.

[16] "This equilibrium then becomes the conventional way of playing the game, because for as long as anyone can remember, the game has always been played in this way. Therefore sampling does not matter any more, because no matter what samples the agents take, their optimal responses will be to play the equilibrium that it already in place" (Young, (1993), 66).

is concentrated around just one equilibrium; this is the state that, in the long run, will be observed with probability close to one.

In order to apply Young's result we have to derive the stochastic potential of the game. We start by observing that a Markov chain can be represented by a tree having a node in each state. A tree rooted at h consists of directed edges such that from every node $h' \neq h$ there exists one and only one direct path from h to h'. Let the *resistance* $r(h, h')$ be the total number of mistakes needed to move from state h to the successor[17] state h'. Let the *total resistance* of a rooted tree to be the sum of resistances associated with its edges and let the stochastic potential of a state h be the least total resistance among all $h-$trees. If the basic game is acyclic[18] and the sample is sufficiently limited[19], then Corollary of Theorem 2 in Young (1993) establishes that a stochastically stable state is a convention with minimum stochastic potential.[20]

Each stochastically stable state is thus a convention $h = (\widehat{s}, ..., \widehat{s})$ in which the same Nash equilibrium profile $\widehat{s} = (\widehat{s}_1, \widehat{s}_2)$ is played m times in succession. If k and m are sufficiently large, the stochastically stable state is unique. In order to detect the stochastically stable state it is sufficient to consider, for each strict Nash equilibrium $\widehat{s}$, all the trees rooted at this equilibrium $\widehat{s}$ (rather than at h), where the resistance on each directed edge now tells us the minimum number of mistakes needed to move the economy from one equilibrium to another. Hence, even if the basic game has many strict Nash equilibria, the adaptive process with mistakes can converge to one of these. This, in turn, gives a theory of equilibrium selection where the strict Nash equilibrium with minimum stochastic potential is the selected one.

As put forward by Binmore, Samuelson and Young (2003), the computation of the minimum stochastic potential is made easier when the game satisfies the bandwagon property. This because the minimum number of mistakes needed to switch from the strict Nash equilibrium $\widehat{s} = (\widehat{s}_1, \widehat{s}_2)$ to the strict Nash equilibrium $\overline{s} = (\overline{s}_1, \overline{s}_2)$ is found when agents by mistake play an action belonging to the profile $\overline{s}$; see Claim 10 in the Appendix.

[17] If state h' is not a successor of state h, then $r(h, h') = \infty$.

[18] Acyclicity means that the best reply graph contains no directed cycles. If G is acyclic, then P^0 converges with probability 1 to a Nash equilibrium state from any starting point, provided that sampling is sufficently incomplete. The no-cycling condition is satisfied for every coordination game. Young (1993) proves that the weaker condition of weak acyclicity is sufficient.

[19] The precise condition is if $k \leq m(L_\Gamma + 2)^{-1}$ where $L(e)$ denote the length of the shortest best reply path originated in the profile e and $L_\Gamma = \max L(e)$.

[20] Alternatively, let a *minimal tree* be a rooted tree with minimum total resistance; then a state is stochastically stable if and only if it is the root of a minimal tree. As explained in Binmore, Samuelson and Young (2003), for a fixed ϵ, the states that will receive larger probabilities of transition in the stationary distribution μ^ϵ are those which are the root of trees whose transition probabilities are relatively large. However, when ϵ becomes arbitrarilly small, transitions involving a large numer of mistakes will become relatively less likely than transitions involving a smaller number of mistakes. We then say that a generic state h will be the stochastically stable state if and only if there is no other state with a tree involving a smaller number of mistakes.

To fix the ideas, consider the game G presented in the previous Section and let $\alpha = 2$, $\beta = 1$ and $x = \frac{1}{3}$. This corresponds to a situation in which agent's 2 provides his maximum optimal effort and this effort is $\frac{1}{3}$. Suppose now that agents have only three strategies available: maximum optimal effort (strategy A): $(e_1, e_2) = \left(\frac{1}{6}, \frac{1}{3}\right)$; half of the maximum optimal effort (strategy B): $(e_1, e_2) = \left(\frac{1}{12}, \frac{1}{6}\right)$; and lastly no effort (strategy C): $(e_1, e_2) = (0, 0)$. The game G has thus the following normal form representation:

	A	B	C
A	$\frac{1}{12}; \frac{1}{9}$	$\frac{1}{36}; \frac{1}{12}$	$-\frac{1}{36}; 0$
B	$\frac{7}{144}; 0$	$\frac{7}{144}; \frac{1}{12}$	$-\frac{1}{144}; 0$
C	$0; -\frac{1}{9}$	$0; -\frac{1}{36}$	$0; 0$

(12)

It is easily verified that the (strict) Nash equilibria (in pure strategies) are the profiles $a = (A; A)$, $b = (B; B)$ and $c = (C; C)$. The profile a is Pareto efficient.[21]

Suppose now that one convention is played m times in succession and that players, by mistake, chose a different action (i.e. an action not belonging to the observed equilibrium profile). This may generate a transition to a new state. Given the bandwagon property, this new state can be reached through two paths, the direct path and the composite path.

Consider the direct path first. To be specific, suppose that the Nash equilibrium c is played m times in succession and that player 1 by mistake plays action B. Specifically, we suppose that he plays B for k' periods and C for the remaining $k - k'$ periods where $k' < k$ and $k < m$. From the best reply correspondence it turns out that $k' = \frac{1}{8}k$ is the minimum number of mistakes (made by agent 1) that are sufficient to shift the economy from c to b. When instead the mistakes are made by agent 2, then the minimum number of mistakes needed to shift the economy from c to b is $k' = \frac{1}{4}k$. Since $\min\left(\frac{1}{8}, \frac{1}{4}\right) = \frac{1}{8}$, the resistance of the direct path considered is $r(c, b) = \frac{1}{8}k$.

Table 1 shows the number of mistakes that each player needs to make in order to generate a transition from i to j where $(i, j) = (a, b, c)$ and $i \neq j$.

	player 1	player 2	min
$a \to b$	$k' = \frac{5}{8}k$	$k' = \frac{1}{4}k$	$\frac{1}{4}k$
$a \to c$	$k' = \frac{7}{8}k$	$k' = \frac{3}{4}k$	$\frac{3}{4}k$
$b \to a$	$k' = \frac{3}{8}k$	$k' = \frac{3}{4}k$	$\frac{3}{8}k$
$b \to c$	$k' = \frac{7}{8}k$	$k' = \frac{3}{4}k$	$\frac{3}{4}k$
$c \to a$	$k' = \frac{3}{8}k$	$k' = \frac{3}{4}k$	$\frac{3}{8}k$
$c \to b$	$k' = \frac{1}{8}k$	$k' = \frac{1}{4}k$	$\frac{1}{8}k$

Table 1

In order to derive the $b-$rooted tree we have also to take into account the direct path from convention a to convention b. In this case b is reached because,

[21] It can be shown that a risk dominates c, b risk dominates c but that risk dominance does not allow to select between a and b.

although agents observe that the Nash equilibrium a was played m times in succession, some agent by mistake plays action B. The resistance of the direct path considered is $r(a,b) = \frac{1}{4}k$. As shown in the second column of Table 2, when direct paths only are considered, the total resistance of the $b-$rooted tree is $r(a,b) + r(c,b) = \frac{3}{8}k$.

Other two $b-$rooted trees are possible, both involving a composite path. In the first, the path followed by economy is $c \to a \to b$. As before, suppose that the convention c is played m times in succession and that one player by mistake plays action A. Specifically, we suppose that he plays A for k' periods and C for the remaining $k - k'$ periods. It turns out that the minimum number of mistakes that are sufficient to shift the economy from c to a are $k' = \frac{3}{8}k$ for agent 1 and $k' = \frac{3}{4}k$ for agent 2. Since $\min\left(\frac{3}{8}, \frac{3}{4}\right) = \frac{3}{8}$, it follows that the resistance of the path is $r(c,a) = \frac{3}{8}k$. Suppose now that once in a for a sufficient long period of time, one player by mistake plays action B. Proceeding as before, it turns out that the minimum number of mistakes that are sufficient to shift the economy from a to b are $k' = \frac{5}{8}k$ for agent 1 and $k' = \frac{1}{4}k$ for agent 2. Since $\min\left(\frac{5}{8}, \frac{1}{4}\right) = \frac{1}{4}$, we have $r(a,b) = \frac{1}{4}k$. Consider now the complete composite path $c \to a \to b$; along this path the total resistance of the tree rooted at b is thus $r(c,a) + r(a,b) = \frac{5}{8}k$.

In the second, the path followed by the economy is $a \to c \to b$. By repeating the same reasoning we find that along this path the total resistance of the tree rooted at b is $r(a,c) + r(c,b) = \frac{7}{8}k$. Table 2 shows all the possible $i-$rooted trees[22] for our game (12).

i	direct path	comp. path 1	comp. path 2	$P(i)$
a	$b \underset{\frac{3}{8}k}{\longrightarrow} a \underset{\frac{1}{8}k}{\leftharpoonup} c$	$b \underset{\frac{3}{4}k}{\longrightarrow} c \underset{\frac{3}{8}k}{\longrightarrow} a$	$c \underset{\frac{1}{8}k}{\longrightarrow} b \underset{\frac{3}{8}k}{\longrightarrow} a$	$\frac{1}{2}k$
b	$a \underset{\frac{1}{4}k}{\longrightarrow} b \underset{\frac{1}{8}k}{\leftharpoonup} c$	$a \underset{\frac{3}{4}k}{\longrightarrow} c \underset{\frac{1}{8}k}{\longrightarrow} b$	$c \underset{\frac{3}{8}k}{\longrightarrow} a \underset{\frac{1}{4}k}{\longrightarrow} b$	$\frac{3}{8}k$
c	$a \underset{\frac{3}{4}k}{\longrightarrow} c \underset{\frac{3}{4}k}{\leftharpoonup} b$	$a \underset{\frac{1}{4}k}{\longrightarrow} b \underset{\frac{3}{4}k}{\longrightarrow} c$	$b \underset{\frac{3}{8}k}{\longrightarrow} a \underset{\frac{3}{4}k}{\longrightarrow} c$	k

Table 2

The stochastic potential of state i is the minimum total resistance over all the $i-$rooted trees; hence, among all the trees rooted at i the stochastic potential of this state identifies the minimal tree. As shown in the last column of Table 2, the stochastic potential of state b is

$$
\begin{aligned}
P(b) &= \min\left[r(a,b) + r(c,b), r(a,c) + r(c,b), r(c,a) + r(a,b)\right] \\
&= \min\left[\left(\tfrac{1}{4} + \tfrac{1}{8}\right)k, \left(\tfrac{3}{4} + \tfrac{1}{8}\right)k, \left(\tfrac{3}{8} + \tfrac{1}{4}\right)k\right] \\
&= \tfrac{3}{8}k
\end{aligned}
$$

[22] For games involving more strict Nash equilibria it is not possible to disentangle between direct and composite paths.

The stochastically stable state is the state with minimum stochastic potential. From inspection of the last column of Table 2 we conclude that

$$\min\left[P\left(a\right), P\left(b\right), P\left(c\right)\right] = P\left(b\right)$$

so that the unique stochastically stable state of the above game is b, that is the strict Nash equilibrium profile $(B; B)$ in which each agent supply half of his optimal maximum effort.[23]

5 Stochastically stable states in the stag-hunt game with production

In this Section we derive the stochastically stable states for the economy described in Section 3. Let assume that this economy is populated by N boundedly rational agents and let N_1 and N_2 be the sub-populations of agents 1 and 2 respectively; in each period, one agent is randomly selected from each sub-population to play the stage-game. Since agents are boundedly rational, they are concerned with their stage-game strategies only. Since Young (1993) results hold for a finite game, in order to apply his approach we have to shift from the game G to a game G_δ (defined below) with a finite strategy set. Here δ is a real number sufficiently small and we interpret $1/\delta$ as a degree of precision with which we measure effort. For smaller and smaller values of δ, since we can discriminate more finely between effort levels, the number of possible actions increases. In the limit as $\delta \to 0$, agents can choose their effort from a continuum of values. We shall consider two cases: in the first effort is a continuous variable (in the sense just specified) while in the second effort is a discrete variable.

5.1 Case 1: effort is (in the limit) a continuous variable

Let $x \leq x^*$ and assume that agents can choose their equilibrium[24] effort levels from the finite and discrete sets S_1 and S_2 respectively[25] where

$$\begin{aligned} S_1 &= \left\{0, \delta, 2\delta, ..., \widehat{e}_1^{\max} - \delta, \widehat{e}_1^{\max}\right\} \cup \left\{e_1^S\right\} \\ S_2 &= \left\{0, \theta\delta, 2\theta\delta, ..., \theta\left(\widehat{e}_1^{\max} - \delta\right), \theta\widehat{e}_1^{\max}\right\} \cup \left\{\theta e_1^S\right\}. \end{aligned} \tag{13}$$

[23] A careful reader should have noticed that in our case there is no composite path with total resistance smaller than the total resistance of the direct path.

[24] Notice that the original action set for the generic agent i is $S_i = \left\{0, ..., \widehat{e}_i^{\max}, ..., e_i^{\max}\right\}$. However, since our game satisfies the bandwagon property, we can exclude all the effort levels greater than the maximum optimal one; these actions will never be a Nash equilibrium and do not alter the resistances of transition between states. Here e_i^S denotes the stochastically stable effort level; we include this in the set of feasible actions in order to derive exact results. See Binmore, Samuelson and Young (2003).

[25] When instead $x \geq x^*$, we have to consider the sets S_1' and S_2', where

$$\begin{aligned} S_1' &= \left\{0, \theta^{-1}\delta, ..., \theta^{-1}\left(\widehat{e}_2^{\max} - \delta\right), \theta^{-1}\widehat{e}_2^{\max}\right\} \cup \left\{\theta^{-1}e_2^S\right\} \\ S_2' &= \left\{0, \delta, ..., \widehat{e}_2^{\max} - \delta, \widehat{e}_2^{\max}\right\} \cup \left\{e_2^S\right\}. \end{aligned}$$

In the limit as $\delta \to 0$, agents can choose their effort from a continuum of values. In this case, our Proposition 5 establishes the existence of a stochastically stable state.

In what follows we denote by G_δ the stage game where the two randomly matched players simultaneously choose their effort from the set S_i given by (13) and receive a payoff given by (5).

Consider two Nash equilibria $e = (e_1, \theta e_1)$ and $\overline{e} = (\overline{e}_1, \theta \overline{e}_1)$ and let e be the initial state. Since the game satisfies the bandwagon property (Lemma 3), in order to derive the resistances it is thus sufficient to analyze the restrict game where the only strategies available are those corresponding to these two equilibria, that is $S_1 = \{e_1, \overline{e}_1\}$ and $S_2 = \{\theta e_1, \theta \overline{e}_1\}$. Two cases are possible: either $\overline{e}_1 > e_1$ or $\overline{e}_1 < e_1$. The former corresponds to a situation in which we exit from the state e to the right while the latter corresponds to a situation in which we exit from the state e to the left. We show in Claim 11 in the Appendix that the resistance of the path $e \to \overline{e}$, with $\overline{e} > e$, is

$$r^{+}(e, \overline{e}) = \begin{cases} \theta^2 \dfrac{(e_1 + \overline{e}_1)}{\alpha(1-x)} k & if \quad x \leq x^* \\ \\ \dfrac{e_1 + \overline{e}_1}{\alpha x} k & if \quad x \geq x^*, \end{cases} \tag{14}$$

while the resistance of the path $e \to \overline{e}$, with $\overline{e} < e$, is

$$r^{-}(e, \overline{e}) = \begin{cases} \left(1 - \dfrac{e_1 + \overline{e}_1}{\alpha x}\right) k & if \quad x \leq x^* \\ \\ \left(1 - \theta^2 \dfrac{e_1 + \overline{e}_1}{\alpha(1-x)}\right) k & if \quad x \geq x^*. \end{cases} \tag{15}$$

From (14) and (15) we notice respectively that $r^{+}(e, \overline{e})$ is an increasing function of e_1 and $\overline{e}_1$ while $r^{-}(e, \overline{e})$ is a decreasing function of e_1 and $\overline{e}_1$. For a given state e, and any value of x, it follows that:

(a) the least resistance on an exit path is found on a direct path leading to the adjacent state, i.e. $\overline{e} = e + \delta = (e_1 + \delta, \theta(e_1 + \delta))$ for (14) and $\overline{e} = e - \delta = (e_1 - \delta, \theta(e_1 - \delta))$ for (15);

(b) let $\Gamma(e)$ denote the following $e-$rooted tree

$$0 \overset{r(0,0)}{\rightarrow} \delta \overset{r(\delta,2\delta)}{\rightarrow} \ldots\ e{-}\delta \overset{r(e-\delta,e)}{\rightarrow} e \underset{r(e+\delta,e)}{\leftarrow} e{+}\delta\ \ldots. \underset{r(\widehat{e}^{\max},\widehat{e}^{\max}-\delta)}{\leftarrow} \widehat{e}^{\max} \tag{16}$$

where each edge is weighted by the least resistance – given by (14) and (15) – involved in the corresponding transition. Let $P(e)$ denote the minimum stochastic potential associated with the generic state e. Then $\Gamma(e)$ is the arborescence with minimum stochastic potential $P(e)$.

As shown in Claim 12 in the Appendix, we can write the stochastic potential associated with a generic state e as[26]

$$\begin{aligned} P(e) &= P(e-\delta) + r(e-\delta, e) - r(e, e-\delta) \\ P(e) &= P(e+\delta) + r(e+\delta, e) - r(e, e+\delta) \end{aligned} \tag{17}$$

Since the game G_δ is acyclic, we know from Young (1993) that it has at least one stochastically stable state, $e_\delta^s = \left(e_1^S, e_2^S\right)$. This is the state which minimizes the stochastic potential over all the possible states, i.e. $e_\delta^s = \arg\min_e P(e)$. Let[27] $0 < e_\delta^s < \widehat{e}^{\max}$; then it must be $P(e_\delta^s) < P(e_\delta^s + \delta)$ and $P(e_\delta^s) < P(e_\delta^s - \delta)$, conditions satisfied when[28]

$$\begin{cases} r(e_\delta^s + \delta, e_\delta^s) - r(e_\delta^s, e_\delta^s + \delta) < 0 \\ r(e_\delta^s - \delta, e_\delta^s) - r(e_\delta^s, e_\delta^s - \delta) < 0. \end{cases} \tag{18}$$

From (14), (15) and (18) it then follows that, for any value of the distributive parameter, e_δ^s is a stochastically stable state if

$$\left| e_\delta^s - \frac{\alpha x}{2} \frac{1-x}{1+x\left(\theta^2-1\right)} \right| < \frac{\delta}{2}.$$

Therefore, as $\delta \to 0$ the stochastically stable state tends to the equilibrium

$$e^s = \left(e_1^S(x), e_2^S(x)\right) = \left(\frac{\alpha x}{2} \frac{(1-x)}{1+x\left(\theta^2-1\right)}, \frac{\alpha x}{2} \frac{\theta(1-x)}{1+x\left(\theta^2-1\right)} \right).$$

Notice that

$$e_1^S = \begin{cases} \widehat{e}_1^{\max} \left(1 - \frac{x\theta^2}{1+x(\theta^2-1)}\right) & if \quad x \le x^* \\ \widehat{e}_2^{\max} \theta^{-1} \left(1 - \frac{1-x}{1+x(\theta^2-1)}\right) & if \quad x \ge x^*, \end{cases} \tag{19}$$

so that, for any x and for any θ, $\widehat{e}_i^{\max} > e_i^S$ since

$$\frac{(1-x)}{1+x(\theta^2-1)} < 1 \quad if \quad x \le x^*$$

$$\frac{x\theta^2}{1+x(\theta^2-1)} < 1 \quad if \quad x \ge x^*.$$

[26] From (14) and (15) we may also compute the resistances of the path $\overline{e} \to e$. When $\overline{e} > e$, $r^-(\overline{e}, e) = r(e+\delta, e)$; when $\overline{e} < e$, $r^+(\overline{e}, e) = r(e-\delta, e)$.

[27] The cases $e_\delta^s = \widehat{e}^{\max}$ and $e_\delta^s = 0$ will be considered below.

[28] As shown above, for any value of x, the resistances $r(e, e+\delta)$ and $r(e, e-\delta)$ are respectively an increasing and a decreasing function of e. Then, if conditions (18) are satisfied, it follows that $P\left(e_\delta^s + (k-1)\delta\right) < P\left(e_\delta^s + k\delta\right)$ and $P\left(e_\delta^s - (k-1)\delta\right) < P\left(e_\delta^s - k\delta\right)$. This ensures that the stochastic potential has a global minimum at e_δ^s.

From Point (b) of Proposition 1 it then follows that the stochastically stable state e^s is not Pareto efficient.

In previous analysis we assumed $0 < e_\delta^s < \widehat{e}^{\max}$. We have now to verify that effectively the lower and upper bound of the action set can not be stochastically stable states. Suppose first that $e_\delta^s = \widehat{e}^{\max}$. Since a state involving an effort grater than the maximum optimal one can not be a Nash equilibrium, it follows that $\widehat{e}^{\max}$ is stochastically stable only if $P\left(\widehat{e}^{\max}\right) < P\left(\widehat{e}^{\max} - \delta\right)$, condition satisfied when

$$r\left(\widehat{e}^{\max} - \delta, \widehat{e}^{\max}\right) - r\left(\widehat{e}^{\max}, \widehat{e}^{\max} - \delta\right) < 0. \tag{20}$$

From (14), (15) we may write (20) as

$$\begin{cases} \frac{\alpha x}{2} < \frac{\alpha x}{2}\frac{1-x}{1+x(\theta^2-1)} + \frac{\delta}{2} & if \qquad x \le x^* \\ \\ \frac{(1-x)\beta^2}{2\alpha} < \frac{\alpha x}{2}\frac{1-x}{1+x(\theta^2-1)} + \frac{\delta}{2} & if \qquad x \ge x^* \end{cases} \tag{21}$$

from which we conclude that $\widehat{e}^{\max}$ is stochastically stable when either $\delta > \delta_1$ and $x \le x^*$ or $\delta > \delta_2$ and $x \ge x^*$, where

$$\begin{aligned} \delta_1 &\equiv \frac{\alpha x^2\theta^2}{(\theta^2 x + (1-x))} \\ \\ \delta_2 &\equiv \frac{\alpha (1-x)^2}{\theta^2(\theta^2 x + (1-x))}. \end{aligned} \tag{22}$$

Since these conditions can not be satisfied as $\delta \to 0$, it follows that $\widehat{e}^{\max}$ can not be a stochastically stable state.

Suppose now that $e_\delta^s = 0$. Since a state involving an effort smaller that zero is not feasible, it follows that zero effort is a stochastically stable state only if $P(0) < P(\delta)$, condition satisfied when

$$r(\delta, 0) - r(0, \delta) < 0. \tag{23}$$

From (14), (15), since

$$r(\delta, 0) - r(0, \delta) = \begin{cases} 1 + \delta\frac{1-x\left(1+\theta^2\right)}{2x(1-x)} & if \qquad x \le x^* \\ \\ 1 + \delta\frac{-1+x\left(1+\theta^2\right)}{2x(1-x)} & if \qquad x \ge x^*, \end{cases}$$

it follows that (23) is never satisfied. We can summarize this discussion in the following Proposition.

Proposition 5 *Let G be the continuous game and let G_δ be a discrete approximation of G with precision $1/\delta$ where $\delta < \delta_1$ for $x \le x^*$ and $\delta < \delta_2$ for $x \ge x^*$.*

Let x be given. As $\delta \to 0$, G_δ has a unique stochastically stable equilibrium given by:

$$\left(e_1^S(x); e_2^S(x)\right) = \left(\frac{\alpha x}{2}\frac{(1-x)}{1+x\left(\theta^2-1\right)}, \theta\frac{\alpha x}{2}\frac{(1-x)}{1+x\left(\theta^2-1\right)}\right). \tag{24}$$

The stochastically stable equilibrium is not Pareto efficient.

Proposition 5 says that when $\delta \to 0$ and for any value of the distributive parameter, the game played by boundedly rational strangers converges to a stochastically stable but Pareto inefficient state. The precise equilibrium to which our economy converges depends on the value of the distributive parameter. When sampling is sufficiently large (although incomplete) and both k/m and the probability of mistakes are sufficiently small, in the long run the equilibrium $\left(e_1^S, e_2^S\right)$ will be observed with the highest positive probability so that it tends to persist and becomes the conventional way of playing the stag hunt game.

To get an idea of how much the stochastically stable levels of effort differ from the maximum optimal levels, consider (19) and notice that $e_1^S(x) = \frac{1}{2}\widehat{e}_1^{\max} = \frac{1}{2}\widehat{e}_2^{\max}\theta^{-1}$ for $x = x^*$ while $\frac{1}{2}\widehat{e}_1^{\max} < e_1^S(x) < \widehat{e}_1^{\max}$ when $x < x^*$ and $\frac{1}{2}\widehat{e}_2^{\max}\theta^{-1} < e_1^S(x) < \widehat{e}_2^{\max}\theta^{-1}$ when $x > x^*$. Analogously for player 2. Therefore, for any x, the stochastically stable effort is not smaller that half of the maximum optimal effort.[29] Notice that when $x \leq x^*$, the lower θ (i.e. α/β), the lower the distance between the stochastically stable equilibrium and the equilibrium with maximal effort. The opposite obtains for $x \geq x^*$.

Substituting (24) into (1) and (2) yields the production and the individual payoffs at the stochastically stable state:

$$\begin{aligned}
Y^S(x) &= \tfrac{\alpha^2 x}{2}\tfrac{(1-x)}{1+x(\theta^2-1)} \\
V_1^S(x) &= \tfrac{\alpha^2 x^2}{4}\tfrac{(1-x)}{(1+x(\theta^2-1))^2}\left(1-x+2x\theta^2\right) \\
V_2^S(x) &= \tfrac{\alpha^2 x}{4}\tfrac{(1-x)^2}{(1+x(\theta^2-1))^2}\left(2(1-x)+x\theta^2\right)
\end{aligned} \tag{25}$$

Example 2. Let us reconsider now previous example 1. Let $\alpha = 2$ and $\beta = 1$. The stochastically stable equilibrium is the strategy profile

$$\left(e_1^S(x), e_2^S(x)\right) = \left(\frac{x-x^2}{3x+1}, \frac{2x-2x^2}{3x+1}\right).$$

The associated level of production is $Y^S(x) = 2x\frac{1-x}{3x+1}$ while agent's payoffs are

$$\left(V_1^S, V_2^S\right) = \left(x^2\frac{1-x}{(3x+1)^2}(7x+1),\ x(1-x)^2\frac{2x+2}{(3x+1)^2}\right).$$

[29] However, letting $V^S(x) = \left(V_1^S(x), V_2^S(x)\right)$ and $V(x) = (V_1(x), V_2(x))$, this does not mean that $\left\|V^S(x) - V(x)\right\|$ is maximum when $x = x^*$.

Figures 4 and 5 plot respectively the total production and the individual utility corresponding to the stochastically stable equilibrium (dotted curves). To facilitate the comparison with previous Example 1, we have also plotted in these Figures the total production and the individual payoffs when agents supply their maximum optimal effort.

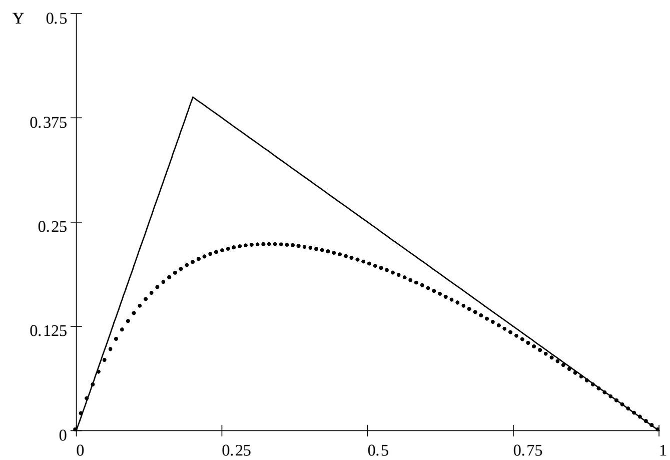

Figure 4 - Total production at the stochastically stable equilibria (dotted curve).

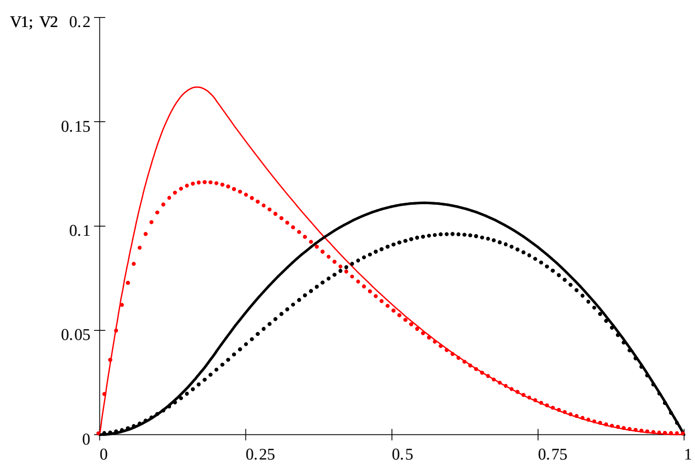

Figure 5 - V_1 and V_2 at the stochastically stable equilibria (dotted curves).

It is evident from (24) and (25) that the stochastically stable equilibrium depends on the distributive parameter. By varying x we obtain a different stochastically stable equilibrium. We can then define the set of stochastically stable equilibria.

Definition 6 *Consider the stochastically stable Nash equilibria in which, for any given x, agents supply the effort levels $\left(e_1^S, e_2^S\right)$. The stochastically stable Utility Distribution Frontier (S-UDF) describes how the corresponding utility pair $\left(V_1^S, V_2^S\right)$ varies with x.*

Figure 6 plots two utility frontiers, both derived for $\alpha = \beta = 2$. The outer locus is the UDF while the S-UDF is represented by the inner locus. In the same Figure we have also plotted three different equilibrium paths $V_2(V_1)$, each derived for a specific value of x. Let $x = x^*$ and consider the corresponding linear path. Suppose that agents supply their optimal maximum effort so that the economy is at the point in which the linear path intersects the UDF. Proposition 5 tells us that this Pareto efficient Nash equilibrium is not stochastically stable. To see why, suppose that in all the past m periods agents played the Nash strategies $(\widehat{e}_1^{\max}, \theta\widehat{e}_1^{\max})$. For any sample of size k they consider, the history of the play instructs our agents to continue to select these strategies. Hence, if agents do not make mistakes, we expect to observe in the long run the Pareto efficient equilibrium. Suppose now that agents do make mistakes. Specifically, let agent 1 choose by mistake $e_1 = \widehat{e}_1^{\max} - \delta < \widehat{e}_1^{\max}$ from periods $t = m+1$ to $t = m+k''$ inclusive, where $k'' \leq k$. If this number of mistakes is appropriate (see (45) in the Appendix), then it induces agent 2 to choose $\theta e_1 = \theta(\widehat{e}_1^{\max} - \delta)$ as his best reply. This, in turn, is sufficient to move the economy from the Pareto efficient equilibrium to the inefficient equilibrium $(\widehat{e}_1^{\max} - \delta, \theta(\widehat{e}_1^{\max} - \delta))$. In terms of Figure 6, this corresponds to a move from the point in which the linear path intersects the UDF to a point on the same path, but below the UDF. Of course, this inefficient equilibrium need not be stochastically stable. Proposition 5 says that a stochastically stable state does exist: even if the probability of making a mistake tends to zero, the fact that agents can do a mistake is sufficient to make the Pareto efficient equilibrium not a stochastically stable state and to drive the economy away from it. In Figure 6, the stochastically stable equilibrium corresponds to the point in which the linear path intersects the S-UDF. At this particular stochastically stable state, each agent supply exactly one half of his maximum optimal effort.

Let $(e_1, e_2) = (0, 0)$ be the state of nature in which no agent provides any effort to the joint project. Proposition 5, far from suggesting that the social contract "might degenerate spontaneously into the state of nature" as claimed by Skyrms (2004; pg. 12), tells us that we can quite confidently expect the emergence of a *minimal* level of social cooperation.

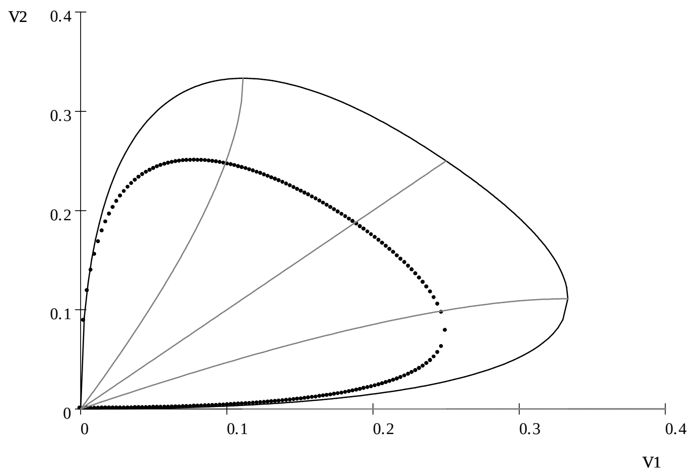

Figure 6 - UDF, S-UDF and equilibrium paths.

This discussion also suggests that it can effectively be difficult for our agents to improve upon the conventional social contract. To see why, let $x = x^*$ and consider the corresponding stochastically stable state, $\left(e_1^S, e_2^S\right)$. Suppose that, although in the past m periods agents played these strategies, agent 1 chooses by mistake[30] $e_1 = e_1^S + \delta > e_1^S$ from periods $t = m+1$ to $t = m + k'$ inclusive, where $k' \leq k$; suppose also that these mistakes can induce agent 2 to choose an higher level of effort as his best reply, i.e. $\theta e_2 = \theta \left(e_1^S + \delta\right) > \theta e_1^S$. It can effectively be the case that these mistakes move the economy from $\left(e_1^S, e_2^S\right)$ to the Pareto efficient equilibrium. However, Proposition 5 says that – since $\left(e_1^S, e_2^S\right)$ is the stochastically stable state — the number of mistakes needed to move the economy from $\left(e_1^S, e_2^S\right)$ to $(\widehat{e}_1^{\max}, \theta\widehat{e}_1^{\max})$ is bigger than the number of mistakes needed for a move in the opposite direction to occur. In other terms, in the long run the probability of observing $(\widehat{e}_1^{\max}, \theta\widehat{e}_1^{\max})$ is smaller than the probability of observing $\left(e_1^S, e_2^S\right)$. This gives a sense in which to improve upon the conventional social contract can be quite hard for our boundedly rational strangers.[31]

[30] As we have seen, since all the equilibria along the path $V_2(V_1)$ corresponding to this particular value of x are possible, the choice of supplying the maximum optimal effort is exposed to the strategic risk of ending up with a lower payoff; this is the case if the other player does not make his part (i.e if he does nor supply his maximal optimal effort).

[31] Of course, the above argument does not imply that it is impossible to improve upon the stochastically stable social contract. Suppose that at the beginning of each stage game, an external observer instructs our strangers to play a cooperative solution. The particular cooperative solution is irrelevant. They could agree to implement the utilitarian distribution, resulting in a distributive parameter $x = x_U$. Alternatively, they could agree to implement the Nash bargaining solution (corresponding to $x = x_N$) or the Rawlsian solution (corresponding to $x = x_R$). The only possiblity for our agents to improve upon the conventional social contract is to sign a conditional contract whose enforcement is assured by the external observer. In this contract they agree (a) on a particular division of the fruits of social cooperation (for instance, $x = x^*$) and (b) to supply their maximal optimal effort, given this value of x. The enforcement is ensured by the external observer who can punish any detected deviation from this contract.

Remark 1. We can use Theorem 2 in Ellison (2000) to show that our economy converges to the stochastically stable state in finite time. Let the radius of the generic state e be the minimum number of mistakes needed to leave this state; in our case $R(e) = \min(r(e, e+\delta), r(e, e-\delta))$. Ellison introduces the concept of modified coradius in order to formalize the observation that a large change will occur more rapidly if it involves a gradual change between consecutive states. Let $r_T(e)$ denote the minimum total resistance over all possible paths from e to e^s. Define the adjusted total resistance, $r_T^*(e)$, by subtracting from $r_T(e)$ the radius of the intermediate states through which the path passes. In our model we have

$$r_T^*(e) = \begin{cases} r(e, e+\delta) & if \quad e < e^s \\ \\ r(e, e-\delta) & if \quad e < e^s. \end{cases}$$

The adjusted coradius CR^* of the stochastically stable equilibrium is the maximum $r_T^*(e)$ over all possible different states. In our model, $r(e, e+\delta)$ is increasing in e while $r(e, e-\delta)$ is decreasing in e; then the maximum value of $r_T^*(e)$ is found when $e = e^s - \delta$ for $e < e^s$ and when $e = e^s + \delta$ for $e > e^s$. Hence $CR^*(e^s) = \max(r(e^s - \delta, e^s), r(e^s + \delta, e^s))$. When $R(e_\delta^s) > CR^*(e_\delta^s)$, Theorem 2 in Ellison (2000) gives some information on the speed of the adjustment. Since in our economy the conditions (18) are satisfied, it then necessarily follows that $R(e_\delta^s) > CR^*(e_\delta^s)$ so that we can apply Ellison's Theorem 2. Let $W(e, e_\delta^s, \epsilon)$ denote the expected wait until a state e^s is first reached from any different state e in the $\epsilon-$perturbed model. Then from Theorem 2, point b) in Ellison (2000) it follows that[32]

$$W(e, e_\delta^s, \epsilon) = O\left(\epsilon^{-CR^*(e_\delta^s)}\right)$$

as $\epsilon \to 0$. Since $CR^*(e^s) = r(e^s - \delta, e^s)$, we have

$$CR^*(e^s) = \begin{cases} \theta^2 \frac{2e^s - \delta}{\alpha(1-x)} & if \quad x < x^* \\ \\ \frac{2e^s - \delta}{\alpha x} & if \quad x > x^*. \end{cases} \tag{26}$$

From (24) and (26) we obtain

$$CR^*(e^s) < \begin{cases} \xi_1 = \theta^2 \frac{x}{1-x+x\theta^2} & if \quad x < x^* \\ \\ \xi_2 = \frac{1-x}{1-x+x\theta^2} & if \quad x > x^* \end{cases}$$

where $\xi_2 = 1 - \xi_1$. Then, for any x, since $\max \xi_1 = \max \xi_2 = \frac{1}{2}$, we get $CR^*(e_\delta^s) < \frac{1}{2}$. Hence there exists a positive constant ϕ such that $W(e, e_\delta^s, \epsilon) < \frac{\phi}{\sqrt{\epsilon}}$.

This solution, however, seems too demanding for our boundedly rational strangers. See also Binmore (1994) for a telling criticism.

[32] Following Ellison, we write $f(z) = O(g(z))$ for $z \to \overline{z}$ as a short-hand for "there exists a constant C such that $f(z)/g(z) = C$ as $z \to \overline{z}$".

Remark 2. Let us consider the following potential function

$$\rho(e_1, e_2) = (1-x)\,x \min[\alpha e_1, \beta e_2] - \left[(1-x)\,e_1^2 + x e_2^2\right].$$

Since

$$\begin{aligned} V_1(e_1, e_2) - V_1(e_1^*, e_2) &= (1-x)^{-1}\left[\rho(e_1, e_2) - \rho(e_1^*, e_2)\right] \\ V_2(e_1, e_2) - V_2(e_1, e_2^*) &= x^{-1}\left[\rho(e_1, e_2) - \rho(e_1, e_2^*)\right] \end{aligned}$$

it then follows from Mondered and Shapley (1996) that the game G is weighted potential game. As suggested by these authors, the equilibrium selection predicted by the maximization of the potential function ρ is the equilibrium that is supported by the experimental evidence provided by Van Huyck et al. (1990). In our case, few computations show that the Nash equilibrium which maximizes the potential function ρ is the stochastically stable one.

5.2 Case 2: effort is a discrete variable

We have seen in previous Section that, when effort is a continuous variable, the stochastically stable state involves minimal cooperation. In this Section we show that when effort can take a finite number of discrete values, it is possible to obtain a stochastically stable state involving maximal cooperation.[33] However this does not necessarily means that our economy converges to a Pareto efficient equilibrium. This occurs for particular values of the distributive parameter only; for other values of the distributive parameter, at the stochastically stable state agents do provide their maximal optimal efforts; however both agents could be better off by altering the value of the distributive parameter.

For illustrative purposes, let us consider the most extreme case in which agents can choose among two strategies (H, L) only, where H coincides with the maximum optimal effort ($\overline{e}$, with $\overline{e} = \min\left(\widehat{e}_1^{\max}, \theta^{-1}\widehat{e}_2^{\max}\right)$), while L coincides with zero effort. Taking (2) into account, we obtain the following payoff matrix corresponding to the traditional stag hunt game in which (H_1, H_2) and (L_1, L_2) are the two pure strategies Nash equilibria:

	H_2	L_2
H_1	$\overline{e}(\alpha x - \overline{e}),\ \overline{e}\left(\alpha(1-x) - \theta^2\overline{e}\right)$	$-\overline{e}^2$, 0
L_1	0, $-\theta^2\overline{e}^2$	0, 0

It can be shown that, for any distributional rule, the Pareto efficient equilibrium is now stochastically stable and risk dominant. The reason is that we have considered the case in which strategic uncertainty is the minimum conceivable.

[33] The experimantal literature showed that coordination failures can effectively arise in a symmetric stag hunt game with a finite number of discrete strategies. This led Cooper (1999) to notice that this type of games are not a mere technical curiosity.

Consider the strategy sets (13) and let $\delta \equiv \frac{1}{n}\widehat{e}_i^{\max}$ where n is a natural number not smaller than 1. Let $x \leq x^*$ and assume that agents can now choose their equilibrium effort levels from the following finite and discrete sets S_1 and S_2[34] where $1 \leq \omega \leq n$:

$$\begin{aligned} S_1 &= \left\{0, \tfrac{1}{n}\widehat{e}_1^{\max}, ..., \tfrac{\omega}{n}\widehat{e}_1^{\max}, ..., \left(\tfrac{n-1}{n}\right)\widehat{e}_1^{\max}, \widehat{e}_1^{\max}\right\} \\ S_2 &= \left\{0, \tfrac{\theta}{n}\widehat{e}_1^{\max}, ..., \tfrac{\omega\theta}{n}\widehat{e}_1^{\max}, ..., \tfrac{\theta(n-1)}{n}\widehat{e}_1^{\max}, \theta\widehat{e}_1^{\max}\right\} \end{aligned} \tag{27}$$

For instance, when $n = 1$, the possible actions are $(0, \widehat{e}_1^{\max})$ for agent 1 and $(0, \theta\widehat{e}_1^{\max})$ for agent 2; when instead $n = 2$, the possible actions are $\left(0, \frac{1}{2}\widehat{e}_1^{\max}, \widehat{e}_1^{\max}\right)$ for agent 1 and $\left(0, \frac{\theta}{2}\widehat{e}_1^{\max}, \theta\widehat{e}_1^{\max}\right)$ for agent 2. And so on.

In what follows we denote by G_n the stage game where the two randomly matched players simultaneously choose their discrete level of effort from the set S_i given by (27) and receive a payoff given by (5).

Let $e^\omega = (e_1^\omega, \theta e_1^\omega)$ be a generic state where $e_1^\omega = \frac{\omega}{n}\widehat{e}_1^{\max}$. Since the analysis of previous Case 1 is still valid, we can derive the following informations.

(a) The least resistance on an exit path is found on a direct path leading to the adjacent state, i.e. $e^{\omega+1} = e + \delta$ and $e^{\omega-1} = e - \delta$; these least resistances are

$$r\left(e^\omega, e^{\omega+1}\right) = \begin{cases} \frac{x\theta^2}{1-x}\frac{\omega+\frac{1}{2}}{n}k & if \quad x \leq x^* \\ \\ \frac{1-x}{x\theta^2}\frac{\omega+\frac{1}{2}}{n}k & if \quad x \geq x^*, \end{cases} \tag{28}$$

and

$$r\left(e^\omega, e^{\omega-1}\right) = \left[1 - \frac{1}{n}\left(\omega - \frac{1}{2}\right)\right]k. \tag{29}$$

(b) The stochastic potential associated with a generic state e^ω is still given by (17) where $e^{\omega+1} = e + \delta$ and $e^{\omega-1} = e - \delta$.

(c) When $\omega = n$, the state e^ω coincides with the equilibrium with maximal cooperation $\widehat{e}^{\max}$. This is stochastically stable if (21) holds where now $\delta = \frac{1}{n}\widehat{e}_1^{\max}$ if $x \leq x^*$ and $\delta = \frac{1}{n}\widehat{e}_2^{\max}$ if $x \geq x^*$. Consider the case $x \leq x^*$. From (21) it follows that $\widehat{e}^{\max} = (\widehat{e}_1^{\max}, \theta\widehat{e}_1^{\max})$ is stochastically stable if

$$\frac{\alpha x}{2} < \frac{\alpha x}{2}\frac{1-x}{1+x\left(\theta^2-1\right)} + \frac{\alpha x}{4n},$$

condition satisfied when

$$0 < x < x_n^{\max} \equiv \frac{1}{\theta^2\left(2n-1\right)+1}, \tag{30}$$

[34] When instead $x \geq x^*$, we obtain the two set

$$\begin{aligned} S_1' &= \left\{0, \tfrac{\theta^{-1}}{n}\widehat{e}_2^{\max}, ..., \tfrac{\omega\theta^{-1}}{n}\widehat{e}_2^{\max}, ..., \tfrac{\theta^{-1}(n-1)}{n}\widehat{e}_2^{\max}, \theta^{-1}\widehat{e}_2^{\max}\right\} \\ S_2' &= \left\{0, \tfrac{1}{n}\widehat{e}_2^{\max}, ..., \tfrac{\omega}{n}\widehat{e}_2^{\max}, ..., \tfrac{(n-1)}{n}\widehat{e}_2^{\max}, \widehat{e}_2^{\max}\right\}. \end{aligned}$$

where $x_n^{\max} \leq x^*$ for $n \geq 1$. Consider now the case $x \geq x^*$. From (21) it follows that $\widehat{e}^{\max} = \left(\theta^{-1}\widehat{e}_2^{\max}, \widehat{e}_2^{\max}\right)$ is stochastically stable if

$$\frac{(1-x)\,\beta^2}{2\alpha} < \frac{\alpha x}{2}\frac{1-x}{1+x\left(\theta^2-1\right)} + \frac{(1-x)\,\beta^2}{4\alpha n},$$

condition satisfied when

$$\frac{2n-1}{\theta^2+(2n-1)} \equiv x_n^{\min} < x < 1, \tag{31}$$

where $x_n^{\min} \geq x^*$ for $n \geq 1$. We can summarize this result in the following Proposition.

Proposition 7 *Let x be given and consider the game G_n. Let $\omega = n$ and consider the Nash equilibrium $\widehat{e}^{\max}$ in which agents supply their maximum optimal effort.*

(a) The Nash equilibrium $\left(\widehat{e}_1^{\max}, \theta\widehat{e}_1^{\max}\right)$ is stochastically stable if $x \in \left(0, x_n^{\max}\right)$, where $x_n^{\max}$ is given by (30).

(b) The Nash equilibrium $\left(\theta^{-1}\widehat{e}_2^{\max}, \widehat{e}_2^{\max}\right)$ is stochastically stable if $x \in \left(x_n^{\min}, 1\right)$, where $x_n^{\min}$ is given by (31).

Proposition 7 tells us that when agents can choose among a finite number of discrete strategies, then the Nash equilibrium with maximal cooperation can be stochastically stable.[35] Proposition 7 however does not tell whether the stochastically stable Nash equilibria with maximal cooperation are Pareto efficient states or not. In Corollary 8 below we show that this occurs only if the number of strategies available is not too and agents have sufficiently different productivities. When instead agents have identical or not too different productivities,

[35] When the distributive parameter does not belong to the specified intervals the stochastically stable state involves less than maximal cooperation. For a given n, the generic state e^ω, with $0 < \omega < n$, is stochastically stable if

$$\begin{cases} r\left(e^{\omega+1}, e^{\omega}\right) - r\left(e^{\omega}, e^{\omega+1}\right) < 0 \\ r\left(e^{\omega-1}, e^{\omega}\right) - r\left(e^{\omega}, e^{\omega-1}\right) < 0. \end{cases}$$

For $x < x^*$, these conditions are satisfied if $x \in \left[x^{\min}(\omega), x^{\max}(\omega)\right]$ where

$$\begin{aligned} x^{\min}(\omega) &\equiv \tfrac{2(n-\omega)-1}{\theta^2(2\omega+1)+2(n-\omega)-1} \\ x^{\max}(\omega) &\equiv \tfrac{2(n-\omega)+1}{\theta^2(2\omega-1)+2(n-\omega)+1}. \end{aligned}$$

It can easily be seen that $x^{\max}(\omega) = x_n^{\max}$ when $\omega = n$. Notice that $x^{\max}(\omega+1) = x^{\min}(\omega)$. Consider now the state with cooperation immediately lower than the maximal one, i.e. $e^\omega = e^{n-1} = \frac{n-1}{n}\widehat{e}^{\max}$. This is a stochastically stable state provided that $x \in \left[x_n^{\max}, x^{\max}(n-1)\right] = \left[x^{\min}(n-1), x^{\max}(n-1)\right]$. Analogously considerations hold for $x > x^*$.

although the equilibrium with maximal cooperation is still a stochastically stable state, it is always inefficient. Lastly we show in Corollary 9 that when each agent can choose more that two actions, the equilibrium with maximal cooperation corresponding to the value of the distributive parameter suggested by the utilitarian unweighted cooperative solution will never be stochastically stable.

Corollary 8 *Let $n > 1$ and let $\overline{e}$ be the the stochastically stable state in which agents supply their maximum optimal effort. Then:*

a) when $\theta < \sqrt{\frac{1}{2}}$, for any $2 \leq n < n_1^*$ with

$$n_1^* (\theta) = int \left(1 + \frac{1}{2\theta^2} \right),$$

there exists an interval $(x_2, x_n^{\max})$ such that for any x belonging to it the stochastically stable state is Pareto efficient;

b) when $\theta > \sqrt{2}$, for any $2 \leq n < n_2^*$ with

$$n_2^* (\theta) = int \left(1 + \frac{\theta^2}{2} \right),$$

there exists an interval $\left(x_n^{\min}, x_1\right)$ such that for any x belonging to it the stochastically stable state is Pareto efficient.

c) when $\sqrt{\frac{1}{2}} < \theta < \sqrt{2}$, then for any $n > 1$ the stochastically stable state is Pareto inefficient.

Proof. See the Appendix.

Corollary 9 *For any θ and for any $n > 1$, let $\overline{e}$ be the equilibrium in which agents supply their maximum effort. If $x = x^*$ then $\overline{e}$ is not a stochastically stable state.*

Proof. See the Appendix.

Example 3. Let $n = 2$ so that agents can choose three strategies (H, M, L) where H coincides with the maximum optimal effort, L coincides with zero effort and M coincides with an effort equal to one half of the maximum optimal effort, $e_i = \frac{1}{2}\widehat{e}_i^{\max}$. Taking (2) into account, we obtain the following payoff matrix in which (H_1, H_2), (M_1, M_2) and (L_1, L_2) are the three pure strategies Nash equilibria:

	H_2	M_2	L_2
H_1	$\overline{e}\left(\alpha x - \overline{e}\right), \overline{e}\left(\alpha\left(1-x\right) - \theta^2\overline{e}\right)$	$\overline{e}\left(\frac{\alpha x}{2} - \overline{e}\right), \frac{\overline{e}}{2}\left(\alpha\left(1-x\right) - \theta^2\frac{\overline{e}}{2}\right)$	$-\overline{e}^2, 0$
M_1	$\frac{\overline{e}}{2}\left(\alpha x - \frac{\overline{e}}{2}\right), \overline{e}\left(\frac{\alpha(1-x)}{2} - \theta^2\overline{e}\right)$	$\frac{\overline{e}}{2}\left(\alpha x - \frac{\overline{e}}{2}\right), \frac{\overline{e}}{2}\left(\alpha\left(1-x\right) - \theta^2\frac{\overline{e}}{2}\right)$	$-\frac{\overline{e}^2}{4}, 0$
L_1	$0, -\theta^2\overline{e}^2$	$0, -\theta^2\frac{\overline{e}^2}{4}$	$0, 0$

From Proposition 7, we know that (H_1, H_2) is the stochastically stable equilibrium if

$$x \in \left(0, \frac{1}{3\theta^2+1}\right) \cup \left(\frac{3}{\theta^2+3}, 1\right).$$

Notice that the stochastically stable equilibrium is also Pareto efficient if it belongs to the decreasing arm of the UDF; this occurs if $x \in [x_2, x_1]$ where x_1 and x_2 are given by (8). Taking this into account it follows that the stochastically stable state is a Pareto efficient Nash equilibrium if

$$x \in \left[\frac{1}{\theta^2+2}, \frac{1}{3\theta^2+1}\right) \cup \left(\frac{3}{\theta^2+3}, \frac{1+\theta^2}{2\theta^2+1}\right].$$

Since however x is given, for some values of θ this set can be empty.[36]

6 Conclusions

In this paper we studied a two agents asymmetric stag hunt game. The model has an infinity of strict, Pareto rankable Nash equilibria. The equilibrium selection problem was solved by appealing to the stochastic stability concept put forward by Young (1993). We proved two main results, both holding for a uniform sample size k. When the action sets are numerable infinite sets, we showed that for any value of the distributive parameter we can expect the emergence of a norm involving less than maximal cooperation. When instead the action sets are finite sets of a particular type (in the sense that each agent can choose his maximum optimal effort and fractions of this), we showed that for some value of the distributive parameter we can expect the emergence of a norm involving maximal cooperation. Two extensions of the model studied in this paper are the subject of some work in progress; in the first we study the case in which agents have different sample sizes while in the second we study the case in which the distributive parameter is made endogenous.

[36] This is exactly the case of game (12). In that case $\theta = 2$ and $x = \frac{1}{3} > \frac{1}{5} = x^*$. For these parameters, since $x \notin \left(\frac{3}{7}, \frac{5}{9}\right]$, the stochastically stable state is not a Pareto efficient equilibrium.

7 Appendix

Proof of Proposition 1

We limit the proof to the sufficiency condition. The necessary condition is derived in the proof of Lemma 3.

Consider agent 1 and suppose $\widehat{e}_1 \leq x\frac{\alpha}{2}$. We have $x\frac{\alpha}{2} \leq (1-x)\frac{\beta^2}{2\alpha}$ if $x \leq x^* \equiv \frac{\beta^2}{\alpha^2+\beta^2}$. Hence, for $x \leq x^*$ agent 2 optimal effort is $\widehat{e}_2 = \frac{\alpha}{\beta}\widehat{e}_1$. Given this, agent 1 has no incentive to change his choice.

Consider agent 2 and suppose $\widehat{e}_2 \leq (1-x)\frac{\beta}{2}$. We have $(1-x)\frac{\beta}{2} \leq x\frac{\alpha^2}{2\beta}$ if $x \geq x^*$. Hence, for $x \geq x^*$ agent 1 optimal effort is $e_1 = \frac{\beta}{\alpha}\widehat{e}_2$. Given this, agent 2 has no incentive to change his choice.

Since for each agent the best reply corresponding to any Nash equilibrium contains only one element, any Nash equilibrium is strict.

Let $x \leq x^*$. At the Nash equilibrium, agents' payoffs are respectively:

$$\begin{aligned} V_1(\widehat{e}_1) &= x\alpha\widehat{e}_1 - (\widehat{e}_1)^2; \\ V_2(\widehat{e}_1) &= (1-x)\alpha\widehat{e}_1 - \left(\frac{\alpha}{\beta}\widehat{e}_1\right)^2. \end{aligned}$$

When $\widehat{e}_1 \leq x\frac{\alpha}{2}$ and $x \leq x^*$, both are increasing functions of $\widehat{e}_1$. Therefore, for any $x \leq x^*$, both players get their maximum equilibrium payoff when $\widehat{e}_1 = x\frac{\alpha}{2}$.

Let $x \geq x^*$. At the Nash equilibrium, agents' payoffs are respectively:

$$\begin{aligned} V_1(\widehat{e}_2) &= x\beta\widehat{e}_2 - \left(\frac{\beta}{\alpha}\widehat{e}_2\right)^2; \\ V_2(\widehat{e}_2) &= (1-x)\beta\widehat{e}_2 - (\widehat{e}_2)^2. \end{aligned}$$

When $\widehat{e}_2 \leq (1-x)\frac{\beta}{2}$ and $x \geq x^*$, both are increasing functions of $\widehat{e}_2$ Therefore, for any $x \geq x^*$, both players get their maximum equilibrium payoff when $\widehat{e}_2 = (1-x)\frac{\beta}{2}$. ■

Proof of Lemma 3

The proof is divided in two parts. In the first we prove that the game is acyclic; in the second we prove that it satisfies the bandwagon property.

1) Acyclicity.

Let us consider a generic profile (e_1, e_2) corresponding to a Nash equilibrium of the game and look at the possible individual best reply paths originated in the given profile.

A) Let $e_2 \geq \frac{\alpha^2 x}{2\beta}$ and $e_1 \geq (1-x)\frac{\beta^2}{2\alpha}$.

Suppose $x \leq x^*$. The possible best reply paths starting from (e_1, e_2) are respectively

$$(e_1, e_2) \underset{G1}{\rightarrow} (\widehat{e}_1^{\max}, e_2) \underset{G2}{\rightarrow} (\widehat{e}_1^{\max}, \theta\widehat{e}_1^{\max})$$

for agent 1 and

$$(e_1, e_2) \underset{G2}{\rightarrow} (e_1, \widehat{e}_2^{\max}) \underset{G1}{\rightarrow} (\widehat{e}_1^{\max}, \widehat{e}_2^{\max}) \underset{G2}{\rightarrow} (\widehat{e}_1^{\max}, \widehat{e}_2^{\max})$$

for agent 2. In order to understand these paths, let (e_1, e_2) be given and consider agent's 1 best reply. This leads to the profile $(\widehat{e}_1^{\max}, e_2)$, where $\widehat{e}_1^{\max} = \frac{\alpha x}{2}$. Given this new profile, and since $x \leq x^*$, agent's 2 best reply leads to the Nash equilibrium profile $(\widehat{e}_1^{\max}, \theta\widehat{e}_1^{\max})$. We have thus derived the first path. Let now (e_1, e_2) be given and consider agent's 2 best reply. This leads to the profile $(e_1, \widehat{e}_2^{\max})$ where $\widehat{e}_2^{\max} = (1-x)\frac{\beta}{2}$. Given this new profile, agent's 1 best reply leads to the profile $(\widehat{e}_1^{\max}, \widehat{e}_2^{\max})$. However, from this last profile, agent's 2 best reply leads to the Nash equilibrium profile $(\widehat{e}_1^{\max}, \widehat{e}_2^{\max})$. We have thus derived the second path.

Suppose $x \geq x^*$ Proceeding as above, we can derive the possible best reply paths starting from (e_1, e_2). These are respectively:

$$(e_1, e_2) \underset{G1}{\rightarrow} (\widehat{e}_1^{\max}, e_2) \underset{G2}{\rightarrow} (\widehat{e}_1^{\max}, \widehat{e}_2^{\max}) \underset{G1}{\rightarrow} \left(\theta^{-1}\widehat{e}_2^{\max}, \widehat{e}_2^{\max}\right)$$

for agent 1 and

$$(e_1, e_2) \underset{G2}{\rightarrow} (e_1, \widehat{e}_2^{\max}) \underset{G1}{\rightarrow} \left(\theta^{-1}\widehat{e}_2^{\max}, \widehat{e}_2^{\max}\right)$$

for agent 2.

Let $L(e)$ denote the length of the shortest path of best reply with origin in e. In the case just analyzed we have $L(e) = 2$.

B) Let $e_2 < \frac{\alpha^2 x}{2\beta}$ and $e_1 \geq (1-x)\frac{\beta^2}{2\alpha}$

The best reply paths for agent 2 are as for previous case A. For player 1 the possible best reply paths originated in (e_1, e_2) are respectively

$$(e_1, e_2) \underset{G1}{\rightarrow} \left(\theta^{-1}e_2, e_2\right) = (\widehat{e}_1, \theta\widehat{e}_1)$$

when $x \leq x^*$ (with $\widehat{e}_1 \leq \widehat{e}_1^{\max}$) and

$$(e_1, e_2) \underset{G1}{\rightarrow} \left\{\theta^{-1}e_2; e_2\right\} \rightarrow \begin{cases} \text{if } e_2 \geq \widehat{e}_2^{\max} \underset{G2}{\rightarrow} \left(\theta^{-1}e_2, \widehat{e}_2^{\max}\right) & \underset{G1}{\rightarrow} \quad \left(\theta^{-1}\widehat{e}_2^{\max}, \widehat{e}_2^{\max}\right) \\ \text{if } e_2 \leq \widehat{e}_2^{\max} \underset{G2}{\rightarrow} \left(\theta^{-1}\widehat{e}_2, \widehat{e}_2\right) & \end{cases}$$

when $x \geq x^*$.

Notice that $L(e) = 1$.

C) Let $e_2 < \frac{\alpha^2 x}{2\beta}$ and $e_1 < (1-x)\frac{\beta^2}{2\alpha}$

The best reply paths for agent 1 are as for previous case B. For player 2 the possible best reply paths originated in (e_1, e_2) are respectively

$$(e_1, e_2) \underset{G2}{\rightarrow} (e_1, \theta e_1) \rightarrow \begin{cases} \text{if } e_1 \geq \widehat{e}_1^{\max} \underset{G1}{\rightarrow} (\widehat{e}_1^{\max}, \theta e_1) & \underset{G2}{\rightarrow} \quad (\widehat{e}_1^{\max}, \theta \widehat{e}_1^{\max}) \\ \text{if } e_1 \leq \widehat{e}_1^{\max} \underset{G1}{\rightarrow} (\widehat{e}_1, \theta \widehat{e}_1) & \end{cases}$$

when $x \leq x^*$ and

$$(e_1, e_2) \underset{G2}{\rightarrow} (e_1, \theta e_1) = \left(\theta^{-1} \widehat{e}_2, \widehat{e}_2\right)$$

when $x \geq x^*$ (where $e_2 \leq \widehat{e}_2^{\max}$).

Notice that $L(e) = 1$.

D) Let $e_2 \geq \frac{\alpha^2 x}{2\beta}$ and $e_1 < (1 - x) \frac{\beta^2}{2\alpha}$

The best reply paths for agent 1 are as for previous case A while those of player 2 are as for previous case C. Notice that $L(e) = 1$. This ends the proof of the first part.

We observe that, since only a (pure strategy) Nash equilibrium profile can be a sink of the best reply graph, the proof of the acyclicity is equivalent to a proof of a necessary condition for the existence of (pure strategies) Nash equilibria.

2) Bandwagon property.

Let $(\widehat{e}_1; \theta \widehat{e}_1)$ be the set of Nash equilibria, where $\widehat{e}_1 \leq \min\left(\frac{\alpha x}{2}; \frac{(1-x)\beta^2}{2\alpha}\right)$. Following Binmore, Samuelson and Young (2003), a sufficient condition for the game to exhibit the bandwagon property is that:

$$\begin{aligned} \Psi_1(\widehat{e}, e) &= V_1(\widehat{e}_1, \theta \widehat{e}_1) - V_1(e_1, \theta \widehat{e}_1) - V_1(\widehat{e}_1, e_2) + V_1(e_1, e_2) \geq 0 \\ \Psi_2(\widehat{e}, e) &= V_2(\widehat{e}_1, \theta \widehat{e}_1) - V_2(e_1, \theta \widehat{e}_1) - V_2(\widehat{e}_1, e_2) + V_2(e_1, e_2) \geq 0 \end{aligned} \tag{32}$$

where $\widehat{e} = (\widehat{e}_1, \theta \widehat{e}_1)$ is any Nash equilibrium of the game and $e = (e_1, e_2)$ is any non-equilibrium strategy profile. Recall that

$$\begin{aligned} V_1(\widehat{e}_1, \theta \widehat{e}_1) &= \alpha \widehat{e}_1 x - \widehat{e}_1^2 \\ V_2(\widehat{e}_1, \theta \widehat{e}_1) &= \alpha \widehat{e}_1 (1 - x) - \theta^2 \widehat{e}_1^2. \end{aligned}$$

The following table give informations on the relevant payoffs:

	$V_1(e_1, \theta \widehat{e}_1)$	$V_2(e_1, \theta \widehat{e}_1)$
if $e_1 \leq \widehat{e}_1$:	$\alpha e_1 x - e_1^2$	$\alpha e_1 (1 - x) - \theta^2 \widehat{e}_1^2$
if $e_1 \geq \widehat{e}_1$:	$\alpha \widehat{e}_1 x - e_1^2$	$\alpha \widehat{e}_1 (1 - x) - \theta^2 \widehat{e}_1^2$

	$V_1(\widehat{e}_1, e_2)$	$V_2(\widehat{e}_1, e_2)$
if $\widehat{e}_1 \leq \theta^{-1} e_2$:	$\alpha \widehat{e}_1 x - \widehat{e}_1^2$	$\alpha \widehat{e}_1 (1 - x) - e_2^2$
if $\widehat{e}_1 \geq \theta^{-1} e_2$:	$\beta e_2 x - \widehat{e}_1^2$	$\beta e_2 (1 - x) - e_2^2$

and lastly

	$V_1(e_1, e_2)$	$V_2(e_1, e_2)$
if $e_1 \leq \theta^{-1}e_2$:	$\alpha e_1 x - e_1^2$	$\alpha e_1 (1-x) - e_2^2$
if $e_1 \geq \theta^{-1}e_2$:	$\beta e_2 x - e_1^2$	$\beta e_2 (1-x) - e_2^2$

Next table summarizes all the possible situations:

	$\Psi_1(\widehat{e}, e)$	$\Psi_2(\widehat{e}, e)$
$\theta^{-1}e_2 < e_1 < \widehat{e}_1$	$\alpha x(\widehat{e}_1 - e_1) > 0$	$\alpha(1-x)(\widehat{e}_1 - e_1) > 0$
$e_1 < \theta^{-1}e_2 < \widehat{e}_1$	$\alpha x\left(\widehat{e}_1 - \theta^{-1}e_2\right)\theta^{-1} > 0$	$\alpha(1-x)\left(\widehat{e}_1 - \theta^{-1}e_1\right) > 0$
$e_1 < \widehat{e}_1 < \theta^{-1}e_2$	0	0
$\theta^{-1}e_2 < \widehat{e}_1 < e_1$	0	0
$\widehat{e}_1 < \theta^{-1}e_2 < e_1$	$\alpha x\left(\theta^{-1}e_2 - \widehat{e}_1\right) > 0$	$\alpha(1-x)\left(\theta^{-1}e_2 - \widehat{e}_1\right) > 0$
$\widehat{e}_1 < e_1 < \theta^{-1}e_2$	$\alpha x(e_1 - \widehat{e}_1) > 0$	$\alpha(1-x)(e_1 - \widehat{e}_1) > 0$

Since all the entries of this table are non negative, $\Psi_1(\widehat{e}, e)$ and $\Psi_2(\widehat{e}, e)$ are non negative as well. This ends the proof.■

Claim 10 *Consider an acyclic and finite game with two players. Let $\widehat{s} = (\widehat{s}_1, \widehat{s}_2)$ be any strict Nash equilibrium of the game and let $s = (s_1, s_2)$ be any different strategy profile. Let the bandwagon property* (32) *be satisfied for any (strict) Nash equilibrium profile $\widehat{s}$ and for any other profile s. Then the transition from the (strict) Nash equilibrium $\widehat{s}$ to the (strict) Nash equilibrium $\overline{s} = (\overline{s}_1, \overline{s}_2)$ involves the minimum number of mistakes if the other player by mistakes chooses strategy $\overline{s}_i$.*

Proof. Since the game is acyclic, we know from Young (1993) that there exists a stochastically stable equilibrium and it coincides with a strict pure strategies Nash equilibrium. Let $\widehat{s}$ be an arbitrary Nash equilibrium. We want to find the minimum number of mistakes that player 2 must make in order to move the economy from $\widehat{s} = (\widehat{s}_1, \widehat{s}_2)$ to the other Nash equilibrium $\overline{s} = (\overline{s}_1; \overline{s}_2)$. Analogous considerations holds when the mistakes are made by agent 1. Let m be the memory size and k be the sample size used by both agents.

Suppose that the economy has been in the state $\widehat{s}$ for a long period of time and consider first the case in which by mistake agent 2 chooses $\overline{s}_2$. Specifically let us suppose that agents 2 choose $\overline{s}_2$ by mistake from period $t = m+1$ to $t = m + k'$ inclusive, where $k' \leq k$. If agent 1 draws a sample that includes these k' choices of $\overline{s}_2$, as well as $k - k'$ choices of $\widehat{s}_2$, then agent 1 deduces that the probability that agent 2 plays $\widehat{s}_2$ is $\sigma_2 = 1 - \frac{k'}{k}$ and that the probability that agent 2 plays $\overline{s}_2$ is $1-\sigma_2 = \frac{k'}{k}$. It then follows that agent 1 is indifferent between $\widehat{s}_1$ and $\overline{s}_1$ if the number of mistakes $\overline{s}_2$ made by agent 2 is

$$k' = A_1 k \equiv \frac{V_1(\widehat{s}_1, \widehat{s}_2) - V_1(\overline{s}_1, \widehat{s}_2)}{V_1(\widehat{s}_1, \widehat{s}_2) - V_1(\overline{s}_1, \widehat{s}_2) + V_1(\overline{s}_1, \overline{s}_2) - V_1(\widehat{s}_1, \overline{s}_2)} k. \tag{33}$$

Consider now the case in which by mistake agent 2 chooses $s_2^* \neq \overline{s}_2$. As before, let us suppose that agents 2 choose s_2^* by mistake from period $t = m+1$ to $t = m+k'$ inclusive, where $k' \leq k$. If agent 1 draws a sample that includes these k' choices of s_2^*, as well as $k-k'$ choices of $\widehat{s}_2$, then agent 1 deduces that the probability that agent 2 plays $\widehat{s}_2$ is $\sigma_2 = 1 - \frac{k'}{k}$ and that the probability that agent 2 plays s_2^* is $1-\sigma_2 = \frac{k'}{k}$. It then follows that agent 1 is indifferent between $\widehat{s}_1$ and $\overline{s}_1$ if if the number of mistakes s_2^* made by agent 2 is

$$k' = B_1 k \equiv \frac{V_1(\widehat{s}_1, \widehat{s}_2) - V_1(\overline{s}_1, \widehat{s}_2)}{V_1(\widehat{s}_1, \widehat{s}_2) - V_1(\overline{s}_1, \widehat{s}_2) + V_1(\overline{s}_1, s_2^*) - V_1(\widehat{s}_1, s_2^*)} k. \tag{34}$$

The number of mistakes involving strategy $\overline{s}_2$ is the minimum if $A_1 < B_1$. Since $\widehat{s}$ is a strict Nash equilibrium, the numerators of A_1 and B_1 are strictly positive. Suppose now that the bandwagon property (32) is satisfied; when referred to player 1, this requires

$$\Psi_1(\widehat{s}, s) = V_1(\widehat{s}_1, \widehat{s}_2) - V_1(s_1, \widehat{s}_2) - V_1(\widehat{s}_1, s_2) + V_1(s_1, s_2) \geq 0.$$

In the case of A_1 we have $\widehat{s} = (\widehat{s}_1, \widehat{s}_2)$ and $s = (\overline{s}_1, \overline{s}_2)$. In the case of B_1 we have $\widehat{s} = (\widehat{s}_1, \widehat{s}_2)$ and $s = (\overline{s}_1, s_2^*)$. Since also the denominator[37] of A_1 and B_1 are non negative, it then follows that $A_1 < B_1$ if

$$\Psi_1(\overline{s}, s) = V_1(\overline{s}_1, \overline{s}_2) - V_1(\widehat{s}_1, \overline{s}_2) - V_1(\overline{s}_1, s_2^*) + V_1(\widehat{s}_1, s_2^*) \geq 0. \tag{35}$$

Since $\overline{s} = (\overline{s}_1, \overline{s}_2)$ is a Nash equilibrium, this condition is satisfied if the bandwagon property holds for $\overline{s} = (\overline{s}_1, \overline{s}_2)$ and $s = (\widehat{s}_1, s_2^*)$. ■

Claim 11 *.Consider the game G_δ and let $e = (e_1, \theta e_1)$ be an arbitrary initial Nash equilibrium.*

(a) The path of exit from e to the right and involving the minimum number of mistakes, is the path leading to the adjacent state $e + \delta = (e_1 + \delta, \theta(e_1 + \delta))$. The resistance of the path $e \to e + \delta$ is

$$r(e, e+\delta) = \begin{cases} \theta^2 \dfrac{(2e_1 + \delta)}{\alpha(1-x)} k & if \quad x \leq x^*; \\ \\ \dfrac{2e_1 + \delta}{\alpha x} k & if \quad x \geq x^*. \end{cases} \tag{36}$$

[37] Since $\overline{e}$ and $\widehat{e}$ are two strict Nash equilibria, it follows that the denominator of A_1 is strictly positive. If the denominator of B_1 is zero, then the condition $A_1 < B_1$ is always satisfied. If instead the denominator of B_1 is positive, the condition $A_1 < B_1$ is satisfied if (35) holds.

(b) The path of exit from e to the left and involving the minimum number of mistakes, is the path leading to the adjacent state $e-\delta=(e_1-\delta,\theta(e_1-\delta))$. The resistance of the path $e\to e-\delta$ is

$$r(e,e-\delta)=\begin{cases}\left(1-\dfrac{2e_1-\delta}{\alpha x}\right)k & if \quad x\leq x^*;\\[2ex] \left(1-\theta^2\dfrac{2e_1-\delta}{\alpha(1-x)}\right)k & if \quad x\geq x^*.\end{cases} \tag{37}$$

Proof. Consider two Nash equilibria $e=(e_1,\theta e_1)$ and $\overline{e}=(\overline{e}_1,\theta\overline{e}_1)$ and let e be the fixed initial state. Since the game satisfies the bandwagon property (Lemma 3), we know from Claim 10 in the Appendix that the path of least resistance from e to $\overline{e}$ is the direct path. We now show e to $\overline{e}$ must be adjacent states.

Since the path of least resistance is a direct path, in order to derive the resistance it is thus sufficient to analyze the restrict game where the only strategies available are those corresponding to these two equilibria, that is $S_1=\{e_1,\overline{e}_1\}$ and $S_2=\{\theta e_1,\theta\overline{e}_1\}$. Two cases are possible: either $\overline{e}_1>e_1$ or $\overline{e}_1<e_1$. The former corresponds to a situation in which we exit from the state e to the right (i.e. such that $\overline{e}_1>e_1$) while the latter corresponds to a situation in which we exit from the state e to the left (i.e. such that $\overline{e}_1<e_1$).

A) Let $\overline{e}_1>e_1$ and consider the following payoff matrix

	θe_1	$\theta\overline{e}_1$
e_1	$e_1(\alpha x-e_1)$, $e_1\left(\alpha(1-x)-\theta^2e_1\right)$	$e_1(\alpha x-e_1)$, $(1-x)\alpha e_1-(\theta\overline{e}_1)^2$
$\overline{e}_1$	$e_1\alpha x-\overline{e}_1^2$, $e_1\left(\alpha(1-x)-\theta^2e_1\right)$	$\overline{e}_1(\alpha x-\overline{e_1})$, $\overline{e}_1\left(\alpha(1-x)-\theta^2\overline{e}_1\right)$

(38)

Let (σ_1,σ_2) be a mixed strategy profile where σ_1 (resp. σ_2) is the probability that agent 1 (resp. 2) plays e_1 (resp. θe_1). The best reply correspondence is

$$\begin{aligned} V_1(e_1,\sigma_2) &\geq V_1(\overline{e}_1,\sigma_2) &\iff \sigma_2 &\geq 1-\frac{\overline{e}_1+e_1}{\alpha x}\\ V_2(\sigma_1,\theta e_1) &\geq V_2(\sigma_1,\theta\overline{e}_1) &\iff \sigma_1 &\geq 1-\theta^2\frac{\overline{e}_1+e_1}{\alpha(1-x)}\end{aligned} \tag{39}$$

Let m be the memory size and k be the sample size used by both agents. Suppose that in the past plays agents 2 choose $\theta\overline{e}_1$ by mistake from period $t=m+1$ to $t=m+k'$ inclusive, where $k'\leq k$. If actual agent 1 draws a sample that includes these k' choices of $\theta\overline{e}_1$, as well as $k-k'$ choices of θe_1, then agent 1 deduce that $\sigma_2=1-\frac{k'}{k}$ and $1-\sigma_2=\frac{k'}{k}$. It then follows from (39) that the minimum numbers of mistakes past agents 2 must make in order to induce actual agent 1 to choose $\overline{e}_1$ as best reply is

$$k' \geq \frac{(\overline{e}_1 + e_1)}{\alpha x} k. \tag{40}$$

In other words, k' mistakes by agent 2 are sufficient to move the economy from e to $\overline{e}$.

Analogously, suppose in the past agents 1 choose $\overline{e}_1$ by mistake from period $t = m+1$ to $t = m + k''$ inclusive, where $k'' \leq k$. If actual agent 2 draws a sample that includes these k'' choices of $\overline{e}_1$, as well as $k - k''$ choices of e_1, then agent 2 deduce that $\sigma_1 = 1 - \frac{k''}{k}$ and $1 - \sigma_1 = \frac{k''}{k}$. It then follows from (39) that the minimum numbers of mistakes past agents 1 must make in order to induce actual agent 2 to choose $\theta\overline{e}_1$ as best reply is

$$k'' \geq \theta^2 \frac{(\overline{e}_1 + e_1)}{\alpha (1-x)} k. \tag{41}$$

In other words, k'' mistakes by agent 1 are sufficient to move the economy from e to $\overline{e}$.

Since (40) and (41) are both increasing functions of $(\overline{e}_1 + e_1)$, and since we are considering the case $\overline{e}_1 > e_1$, it follows that the number of mistakes that is sufficient to displace the economy from e to $\overline{e}$ is minimized when $\overline{e}_1 = e_1 + \delta$. Therefore, when the game is in state e, the path of exit (from this state) to the right (i.e. such that $\overline{e}_1 > e_1$) with the minimum number of mistakes is the path leading to the state $\overline{e} = e + \delta = (e_1 + \delta, \theta (e_1 + \delta))$. The resistance in going from e to $e + \delta$ is the minimum number of mistakes sufficient to shift the economy from the first equilibrium to the second one; since the minimum between (40) and (41) depends on whether x is greater or smaller than x^*, we have

$$r(e, e+\delta) = \begin{cases} \theta^2 \dfrac{(2e_1 + \delta)}{\alpha(1-x)} k & if \quad x \leq x^*; \\ \\ \dfrac{2e_1 + \delta}{\alpha x} k & if \quad x \geq x^*. \end{cases}$$

B). Let $\overline{e}_1 < e_1$ and consider the following payoff matrix

	θe_1	$\theta \overline{e}_1$
e_1	$e_1 (\alpha x - e_1); e_1 \left(\alpha (1-x) - \theta^2 e_1\right)$	$\alpha x \overline{e}_1 - e_1^2; (1-x)\alpha \overline{e}_1 - (\theta \overline{e}_1)^2$
$\overline{e}_1$	$\overline{e}_1 \alpha x - \overline{e}_1^2; \overline{e}_1 \alpha (1-x) - (\theta e_1)^2$	$\overline{e}_1 (\alpha x - \overline{e}_1); \overline{e}_1 \left(\alpha(1-x) - \theta^2 \overline{e}_1\right)$

(42)

Let (σ_1, σ_2) be a mixed strategy profile where σ_1 (resp. σ_2) is the probability that agent 1 (resp. 2) plays e_1 (resp. θe_1). The best reply correspondence is

$$\begin{aligned} V_1(e_1, \sigma_2) &\geq V_1(\overline{e}_1, \sigma_2) \iff \sigma_2 \geq \frac{\overline{e}_1 + e_1}{\alpha x} \\ V_2(\sigma_1, \theta e_1) &\geq V_2(\sigma_1, \theta \overline{e}_1) \iff \sigma_1 \geq \theta^2 \frac{\overline{e}_1 + e_1}{\alpha(1-x)} \end{aligned} \tag{43}$$

Proceeding as before, it follows from (43) that the minimum numbers of mistakes past agent 2 must make in order to induce actual agent 1 to choose $\overline{e}_1$ as best reply is

$$k' \geq \left(1 - \frac{(\overline{e}_1 + e_1)}{\alpha x}\right) k \tag{44}$$

while the minimum numbers of mistakes past agent 1 must make in order to induce actual agent 2 to choose $\theta \overline{e}_1$ as best reply is

$$k'' = \left(1 - \theta^2 \frac{(\overline{e}_1 + e_1)}{\alpha (1-x)}\right) k. \tag{45}$$

As before, k' mistakes by agent 2 or k'' mistakes by agent 1 are sufficient to move the economy from e to $\overline{e}$.

Since (44) and (45) are both decreasing functions of $(\overline{e}_1 + e_1)$, and since we are considering the case $\overline{e}_1 < e_1$, it follows that the number of mistakes that is sufficient to displace the economy from e to $\overline{e}$ is minimized when $\overline{e}_1 = e_1 - \delta$. Therefore, when the game is in state e, the path of exit (from this state) to the left (i.e. such that $\overline{e}_1 < e_1$) with the minimum number of mistakes is the path leading to the state $\overline{e} = e - \delta = (e_1 - \delta, \theta (e_1 - \delta))$. The resistance in going from e to $e - \delta$ is the minimum number of mistakes sufficient to shift the economy from the first equilibrium to the second one; as before, since the minimum between (44) and (45) depends on whether x is greater or smaller than x^*, we have

$$r(e, e-\delta) = \begin{cases} \left(1 - \dfrac{2e_1 - \delta}{\alpha x}\right) k & if \quad x \leq x^*; \\ \\ \left(1 - \theta^2 \dfrac{2e_1 - \delta}{\alpha (1-x)}\right) k & if \quad x \geq x^*. \end{cases}$$

This ends the proof. ■

Claim 12 *Consider the game G_δ. Since the $e-$rooted tree with minimum stochastic potential $P(e)$ is $\Gamma(e)$ where*

$$0 \overset{r(0,0)}{\rightarrow} \delta \overset{r(\delta,2\delta)}{\rightarrow} \ldots \; e-\delta \overset{r(e-\delta,e)}{\rightarrow} e \underset{r(e+\delta,e)}{\leftarrow} e+\delta \; \ldots. \underset{r(\widehat{e}^{\max},\widehat{e}^{\max}-\delta)}{\leftarrow} \widehat{e}^{\max},$$

then

$$\begin{aligned} P(e+\delta) &= P(e) + r(e, e+\delta) - r(e+\delta, e) \\ P(e-\delta) &= P(e) + r(e, e-\delta) - r(e-\delta, e). \end{aligned}$$

Proof. Consider, without loss of generality, the following $e-$rooted tree:

$$e - 2\delta \overset{r_1}{\rightarrow} e - \delta \overset{r_2}{\rightarrow} e \overset{r_3}{\leftarrow} e + \delta \overset{r_4}{\leftarrow} e + 2\delta,$$

where $r_1 \equiv r(e-2\delta, e-\delta)$, $r_2 \equiv r(e-\delta, e)$, $r_3 \equiv r(e+\delta, e)$ and $r_4 \equiv r(e+2\delta, e+\delta)$. We have

$$P(e) = r_1 + r_2 + r_3 + r_4. \tag{46}$$

Consider now the following $(e+\delta)$ −rooted tree:

$$e-2\delta \overset{r_1}{\to} e-\delta \overset{r_2}{\to} e \overset{r_3'}{\to} e+\delta \overset{r_4}{\leftarrow} e+2\delta,$$

where $r_3' \equiv r(e, e+\delta)$. We then have

$$P(e+\delta) = r_1 + r_2 + r_3' + r_4. \tag{47}$$

From (46) and (47) we have

$$\begin{aligned} P(e+\delta) &= r_1 + r_2 + r_3' + r_4 + r_3 - r_3 \\ &= P(e) + r_3' - r_3 \\ &= P(e) + r(e, e+\delta) - r(e+\delta, e). \end{aligned}$$

Lastly, consider the following $(e-\delta)$ −rooted tree:

$$e-2\delta \overset{r_1}{\to} e-\delta \overset{r_2'}{\leftarrow} e \overset{r_3}{\leftarrow} e+\delta \overset{r_4}{\leftarrow} e+2\delta,$$

where $r_2' \equiv r(e, e-\delta)$. We then have

$$P(e-\delta) = r_1 + r_2' + r_3 + r_4. \tag{48}$$

From (46) and (48) we have

$$\begin{aligned} P(e-\delta) &= r_1 + r_2' + r_3 + r_4 + r_2 - r_2 \\ &= P(e) + r_2' - r_2 \\ &= P(e) + r(e, e-\delta) - r(e-\delta, e). \end{aligned}$$

■

Proof of Corollary 8

Recall that when $x \leq x^*$, the UDF is an increasing function for $0 \leq x \leq x_2$ and a decreasing function for $x_2 < x \leq x^*$; when instead $x \geq x^*$, the UDF is a decreasing function for $x^* < x \leq x_1$ and an increasing function for $x_1 < x \leq 1$ where x_2 and x_1 are both given in (8).

Let $n > 1$. Then $\overline{e}$ is the stochastically stable state when either $x \in (0, x_n^{\max}) \subset (0, x^*)$ or $x \in \left(x_n^{\min}, 1\right) \subset (x^*, 1)$. Notice that $x_2 < x_n^{\max}$ when $n < n_1^*(\theta)$ while $x_n^{\min} < x_1$ when $n < n_2^*(\theta)$ where

$$\begin{aligned} n_1^*(\theta) &= int\left(1 + \tfrac{1}{2\theta^2}\right) \\ n_2^*(\theta) &= int\left(1 + \tfrac{\theta^2}{2}\right). \end{aligned}$$

Notice that $2 < n_1^*(\theta)$ if $\theta < \sqrt{\frac{1}{2}}$ and $2 < n_2^*(\theta)$ if $\theta > \sqrt{2}$.

Let suppose $\theta \leq \sqrt{\frac{1}{2}}$. Then $n_2^*(\theta) < 2 \leq n_1^*(\theta)$.

(a) When $n \geq n_1^*(\theta)$ then $\overline{e}$ belongs to the increasing arm of the UDF for any $x \in (0, x_n^{\max})$ and $x \in \left(x_n^{\min}, 1\right)$;

(b) When $n < n_1^*(\theta)$ then $\overline{e}$ belongs to the increasing arm of the UDF for any $x \in (0, x_2)$ and $x \in \left(x_n^{\min}, 1\right)$; $\overline{e}$ belongs to the decreasing arm of the UDF for any $x \in (x_2, x_n^{\max})$.

Let suppose $\theta \geq \sqrt{2}$. Then $n_1^*(\theta) < 2 \leq n_2^*(\theta)$.

(a) When $n \geq n_2^*(\theta)$ then $\overline{e}$ belongs to the increasing arm of the UDF for any $x \in (0, x_n^{\max})$ and $x \in \left(x_n^{\min}, 1\right)$;

(b) When $n < n_2^*(\theta)$ then $\overline{e}$ belongs to the increasing arm of the UDF for any $x \in (0, x_n^{\max})$ and $x \in (x_1, 1)$; $\overline{e}$ belongs to the decreasing arm of the UDF for any $x \in \left(x_n^{\min}, x_1\right)$.

Let suppose $\sqrt{\frac{1}{2}} < \theta < \sqrt{2}$. Then $n_2^*(\theta) < 2$ and $n_1^*(\theta) < 2$. Since $n > 1$, then $\overline{e}$ belongs to the increasing arm of the UDF for any $x \in (0, x_n^{\max})$ and $x \in \left(x_n^{\min}, 1\right)$. ■

Proof of Corollary 9.

From Corollary 8, by noting that when $n > 1$, then for any values of θ we get $x_n^{\max} < x^* < x_n^{\min}$. ■

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