Degenerate Feedback and Time Consistency in Dynamic Games

Roberto Cellini,* Luca Lambertini# and George Leitmann§

* Dipartimento di Economia e Metodi Quantitativi

Università di Catania, Corso Italia 55, 95129 Catania, Italy

phone 39-095375344, fax 39-095-370574, cellini@unict.it

Dipartimento di Scienze Economiche, Università di Bologna

Strada Maggiore 45, 40125 Bologna, Italy

phone 39-051-2092600, fax 39-051-2092664, lamberti@spbo.unibo.it

ENCORE, Department of Economics, University of Amsterdam

Roetersstraat 11, WB1018 Amsterdam, The Netherlands

§ College of Engineering, University of California

Berkeley, California 94720, USA, gleit@uclink.berkeley.edu

February 18, 2005

Abstract

This paper analyses the time consistency of open-loop equilibria, in the cases of Nash and Stackelberg behaviour. We define a class of games where the strong time-consistency of the open-loop Nash equilibrium associates with the time consistency of the open-loop Stackelberg equilibrium. We label these games as 'perfect uncontrollable' and provide two examples based on (i) a model where firms invest so as to increase consumers' reservation prices, based upon Cellini and Lambertini (CEJOR, 2003); and (ii) a model where firms compete to increase their respective market shares, based upon Leitmann and Schmitendorf (IEEE Transactions on Automatic Control, 1978).

JEL classification: C72, C73

Keywords: differential games, open-loop equilibria, time consistency, subgame perfection

1 Introduction

The existing literature on simultaneous-move differential games devotes a considerable amount of attention to identifying classes of games where either the feedback or the closed-loop equilibria degenerate into open-loop equilibria. The degeneration means that the Nash-equilibrium time paths of the control variables coincide under the different solution concepts, that is, the optimal paths of control variables depend only on time (and not on states) also under the closed-loop or feedback solution concepts. The interest in the coincidence between the equilibrium path under the different solution concepts is motivated by the following reason. Whenever an open-loop equilibrium is a degenerate closed-loop equilibrium, then the former is also strongly time consistent (or Markovian, or subgame perfect). Therefore, one can rely upon the open-loop equilibrium which, in general, is much easier to derive than closed-loop or feedback ones. Classes of games where this coincidence arises are illustrated in Clemhout and Wan (1974); Reinganum (1982); Mehlmann and Willing (1983); Dockner *et al.* (1985); Fershtman (1987); Fershtman, Kamien and Muller (1992).[1]

Unlike open-loop Nash games, which always generate time consistent (although only weakly) equilibria, Stackelberg open-loop games usually generate time inconsistent equilibria. This has been known ever since Simaan and Cruz (1973a,b).[2] By this, it is meant that, at any time during the game, the leader finds it optimal to modify the plan chosen at the outset. This has yielded a large body of literature with economic applications, in particular those concerned with the time inconsistency of optimal monetary or fiscal policy.[3] In a recent contribution, Xie (1997) singled out a property ensuring the time consistency of open-loop Stackelberg equilibria, labelling such

[1]For an overview, see Mehlmann (1988) and Dockner *et al.* (2000, ch. 7).

[2]See Başar and Olsder (1982, 1995^2, ch. 7) and Dockner *et al.* (2000, ch. 5) for exhaustive overviews.

[3]This literature is too wide to be duly accounted for here. See Persson and Tabellini (2000) for an account.

games as *uncontrollable*, as the leader cannot manipulate the equilibrium at will through his/her control variable.

A striking feature of these two lines of research - the time consistency of open-loop Nash equilibria and the time consistency of open-loop Stackelberg equilibria - is that, so far, they haven't overlapped at all. That is, researchers have looked either for those classes of games yielding strongly time consistent open-loop solutions, or for those yielding time consistent Stackelberg open-loop solutions, but not for both at the same time. This may depend upon the fact that the two solutions are conceptually different. Nevertheless, both for technical reasons and for possible applications, it would be desirable to identify games enjoying both properties at the same time.

This is the aim of the present paper. In section 2, we set out with a general illustration of the problem at stake. We label as *perfect uncontrollable game* the game producing (i) strongly time consistent open-loop Nash equilibria *and* (ii) time consistent open-loop Stackelberg equilibria. In section 3 we illustrate a duopoly game with advertising based upon Cellini and Lambertini (2003) to provide one such example. This game is state linear and the Hamiltonian of each player is additively separable in state and control variables. These two properties suffice to make it a perfect uncontrollable game. Then, in section 4 we proceed to investigate the features of an alternative advertising game (based on Leitmann and Schmitendorf, 1978; and Feichtinger, 1983) where such additive separability does not hold. Still, we prove that all of its open-loop equilibria are indeed degerate feedback ones. Possible extensions and concluding comments are outlined in section 5.

2 Setup

Consider a generic differential game, played over continuous time, with $t \in [0, \infty)$.[4] The set of players is $\mathbb{P} \equiv \{1, 2\}$. Moreover, let $x_i(t)$ and $u_i(t)$ define

[4]One could also consider a finite terminal time T. The specific choice of the horizon is immaterial to the ensuing analysis, if terminal conditions are appropriately defined.

the state variable and the control variable pertaining to player i, while $\mu_{ij}(t)$ is the co-state variable attached by player i to state variable $x_j(t)$, $i, j = 1, 2$. Assume there exists a prescribed set $\mathcal{U}_i$ such that any admissible action $u_i(t) \in \mathcal{U}_i$. The dynamics of player i's state variable is:

$$\frac{dx_i(t)}{dt} \equiv \dot{x}_i(t) = f_i(\mathbf{x}(t), \mathbf{u}(t)) \tag{1}$$

where $\mathbf{x}(t) = (x_1(t), x_2(t))$ is the vector of state variables at time t, and $\mathbf{u}(t) = (u_1(t), u_2(t))$ is the vector of players' actions at the same date, i.e., it is the vector of control variables at time t. That is, in the most general case, the dynamics of player i's state variable depends on all states and controls associated with all players involved in the game. The value of the state variables at $t = 0$ is known: $\mathbf{x}(0) = (x_1(0), x_2(0))$.

Each player has an objective function, defined as the discounted value of the flow of payoffs over time. The instantaneous payoff depends upon the choices made by player i as well as its rivals, that is:

$$\pi_i \equiv \pi_i(\mathbf{x}(t), \mathbf{u}(t)). \tag{2}$$

Given $u_j(t)$, $j \neq i$ and the discount rate ρ, player i's objective is:

$$\max_{u_i(\cdot)} J_i \equiv \int_0^\infty \pi_i(\mathbf{x}(t), \mathbf{u}(t)) e^{-\rho t} dt \tag{3}$$

subject to (1), $u_i(t) \in \mathcal{U}_i$ and initial conditions $\mathbf{x}(0) = (x_1(0), x_2(0))$.

In the literature on differential games, one usually refers to the concepts of *weak* and *strong* time consistency. The difference between these two properties can be outlined as follows:

Definition 1: weak time consistency Consider a game played over $t = [0, \infty)$ and examine the trajectories of the state variables, denoted by $\mathbf{x}(t)$. The equilibrium is weakly time consistent if its truncated part in the time interval $t = [T, \infty)$, with $T \in (0, \infty)$, represents an equilibrium also for any subgame starting from $t = T$, *and* from the vector of initial conditions $\mathbf{x}_T = \mathbf{x}(T)$.

Definition 2: strong time consistency Consider a game played over $t = [0, \infty)$. The equilibrium is strongly time consistent, if its truncated part is an equilibrium for the subgame, independently of the conditions regarding state variables at time T, $\mathbf{x}(T)$.

Strong time consistency requires the ability on the part of each player to account for the rival's behaviour at any point in time, i.e., it is, in general, an attribute of closed-loop equilibria, and corresponds to subgame perfectness. Weak time consistency is a milder requirement and does not ensure, in general, that the resulting Nash equilibrium be subgame perfect.[5]

Now consider the Stackelberg differential game, and assume player i is the follower. The Hamiltonian of player i is:

$$\begin{aligned}\mathcal{H}_i(\mathbf{x}(t), \mathbf{u}(t)) \equiv\ & e^{-\rho t}\left[\pi_i(\mathbf{x}(t), \mathbf{u}(t)) + \lambda_{ii}(t) \cdot f_i(\mathbf{x}(t), \mathbf{u}(t)) + \right. \\ & \left. + \lambda_{ij}(t) \cdot f_j(\mathbf{x}(t), \mathbf{u}(t))\right] , \qquad (4)\end{aligned}$$

where $\lambda_{ij}(t) = \mu_{ij}(t)e^{\rho t}$ is the co-state variable (evaluated at time t) that firm i associates with the state variable $x_j(t)$.

In the remainder of the paper, we shall focus on first order conditions alone, under the assumption that sufficiency conditions are met. This will be apparent in the examples illustrated in the next sections. Moreover, we shall adopt the following conventional notation: steady state values of controls and states are identified by superscript s; optimal controls or states satisfying the necessary conditions are starred.

The first order condition (FOC) on the control variable $u_i(t)$ is:[6]

$$\frac{\partial \mathcal{H}_i(\mathbf{x}^*(t), \mathbf{u}^*(t))}{\partial u_i} = 0,\ i,\ j = 1, 2; \qquad (5)$$

$$\Leftrightarrow \frac{\partial \pi_i(\mathbf{x}^*(t), \mathbf{u}^*(t))}{\partial u_i} + \lambda_{ii}(t)\frac{\partial f_i(\mathbf{x}^*(t), \mathbf{u}^*(t))}{\partial u_i} + \lambda_{ij}(t)\frac{\partial f_j(\mathbf{x}^*(t), \mathbf{u}^*(t))}{\partial u_i} = 0$$

[5] For a more detailed analysis of these issues, see Dockner *et al.* (2000, Section 4.3, pp. 98-107); see also Başar and Olsder (1982, 1995[2], ch. 6).

[6] The indication of exponential discounting is omitted for brevity.

and the adjoint equations concerning the dynamics of state and co-state variables are as follows:

$$-\frac{\partial \mathcal{H}_i\left(\mathbf{x}^*(t), \mathbf{u}^*(t)\right)}{\partial x_j} = \frac{\partial \lambda_{ij}(t)}{\partial t} - \rho \lambda_{ij}\left(t\right) \ , \ i, \ j = 1, 2; \tag{6}$$

$$\Leftrightarrow \quad \frac{\partial \lambda_{ij}(t)}{\partial t} = \rho \lambda_{ij}\left(t\right) - \frac{\partial \pi_i\left(\mathbf{x}^*(t), \mathbf{u}^*(t)\right)}{\partial x_j} + \\ -\lambda_{ii}(t)\frac{\partial f_i\left(\mathbf{x}^*(t), \mathbf{u}^*(t)\right)}{\partial x_j} - \lambda_{ij}(t)\frac{\partial f_j\left(\mathbf{x}^*(t), \mathbf{u}^*(t)\right)}{\partial x_j}$$

They have to be considered alongside with initial conditions $\mathbf{x}(0) = \left(x_1\left(0\right), x_2\left(0\right)\right)$ and transversality conditions, which set the final value (at $t = \infty$) of the state and/or co-state variables. In problems defined over an infinite time horizon, one sets $\lim_{t\to\infty} \ e^{-\rho t}\lambda_{ij}(t)x_j(t) = 0, \ i, j = 1, 2.$[7]

For simplicity, we consider the case where only one state and one control are associated with every single player.

Before proceeding further, it is worth clarifying that there exist two possible approaches to the description of strategic interaction between players in a Stackelberg game. The first consists in asking the leader (player j) to choose a proper reaction function $u_j^*\left(u_i^*\left(t\right)\right)$ specifying the leader's best reply to the follower's behaviour at any t during the game. Alternatively, following Dockner *et al.* (2000, chapter 5.3), one may suppose that the leader can announce to the follower the policy $u_i^L\left(\mathbf{x}(t)\right)$ that she (the leader) will use throughout the game, defined in terms of the states only. The follower, taking $u_i^L\left(\mathbf{x}(t)\right)$ as given, determines the reaction function $u_i^F\left(u_i^L\left(\mathbf{x}(t)\right)\right)$ to maximise his payoff. The problem of the leader is then to choose, among all the admissible rules $u_i\left(\mathbf{x}(t)\right)$, that particular $u_i^L\left(\mathbf{x}(t)\right)$ that maximises her payoff, given the follower's best reply and all the additional constraints. Dockner *et al.* (2000, chapter 5.3) provide the solution to this problem by confining their attention to games where only one state variable appears and therefore one

[7]Otherwise, if $t \in [0, T]$, a different transversality condition applies. For instance, if the value of $x_j\left(T\right)$ differs from zero, then one may set $\lambda_{ij}\left(T\right) = 0$. See section 3.

can write $u_i^L = w + zx$, where $w, z \in \mathbb{R}$. The leader's optimization problem amounts to choosing w and z once and for all. A major objection to this approach is that, indeed, this is not a game where both players have Markovian state information. Moreover, the fact that the leader chooses w and z at the outset and keeps them constant throughout the game is responsible for the time consistency characterising the resulting Stackelberg equilibrium (for a more extensive discussion, see Dockner *et al.*, 2000, p. 135).

In the present paper, we take the first route. From Simaan and Cruz (1973a,b), we know that, in general, open-loop Stackelberg games yield time inconsistent equilibria. However, if the game structure satisfies some specific requirements, the Stackelberg open-loop equilibrium is indeed time consistent (and possibly even strongly so). To illustrate such requirements, one has to proceed as follows. From (5) one obtains the instantaneous best reply of player i, which can be differentiated with respect to time to yield the dynamics of $u_i(t)$. Moreover, given (1), the first order condition (5) will contain the co-state variable $\lambda_{ii}(t)$ associated with the kinematic equation of the state $x_i(t)$. Therefore, (5) can be solved w.r.t. $\lambda_{ii}(t)$ so as to yield:

$$\lambda_{ii}(t) = -\left[\frac{\partial \pi_i\left(\mathbf{x}^*(t), \mathbf{u}^*(t)\right)}{\partial u_i} + \lambda_{ij}(t)\frac{\partial f_j\left(\mathbf{x}^*(t), \mathbf{u}^*(t)\right)}{\partial u_i}\right] / \frac{\partial f_i\left(\mathbf{x}^*(t), \mathbf{u}^*(t)\right)}{\partial u_i} \tag{7}$$

If the r.h.s. in (7) contains the leader's control $u_j(t)$, then the open-loop Stackelberg strategies are time inconsistent, in that the leader controls the follower's state dynamics by manoeuvring $u_j(t)$. In such a case, the game is *controllable* by the leader, who reneges initial plans at any later instant in order to profitably re-optimise. If, instead, $\lambda_{ii}(t)$ does not depend on $u_j(t)$, the game is *uncontrollable* by the leader, and the resulting open-loop Stackelberg equilibrium strategies are time consistent (this definition dates back to Xie, 1997; it is also used by Dockner *et al.*, 2000, ch. 5). I.e., uncontrollability relies on the following property:

$$\frac{\partial \lambda_{ii}(t)}{\partial u_j(t)} = 0,\ i,\ j = 1, 2;\ j \neq i \tag{8}$$

which must hold for (7); likewise, it must be $\partial\lambda_{ij}(t)/\partial u_j(t) = 0$. This amounts to saying that $\lambda_{ii}(t)$ does not depend on $u_j(t)$, i.e., the leader cannot affect the co-state variable of the follower, and the resulting Stackelberg solution is time consistent. Since $\lambda_{ii}(t)$ in (7) comes from the solution of (5), we can say that (8) is equivalent to:

$$\frac{\partial^2 \mathcal{H}_i\left(\mathbf{x}^*(t), \mathbf{u}^*(t)\right)}{\partial u_i \partial u_j} = 0,\ i,\ j = 1, 2;\ j \neq i. \tag{9}$$

For completeness, we may briefly summarise the issue of strong time consistency of open-loop Nash equilibria. If

$$\frac{\partial^2 \mathcal{H}_i\left(\mathbf{x}^*(t), \mathbf{u}^*(t)\right)}{\partial u_i \partial x_j} = 0,\ i,\ j = 1, 2;\ j \neq i \tag{10}$$

either immediately from the Hamiltonian function, or by substitutions from the co-state equations into first order conditions, then the optimal controls are independent of states and the open-loop equilibrium is subgame perfect since it is strongly time consistent. The property whereby FOCs on controls are independent of states and initial conditions after replacing the optimal values of the co-state variables is known as *state-redundancy*, and the game itself as *state-redundant* or *perfect* (Fershtman, 1987; Mehlmann, 1988, ch. 4). Of course condition (9) in general does not coincide with (10). Whenever (9) and (10) are simultaneously met within the same game, then the game itself is a *perfect uncontrollable game*. Accordingly, we may formulate:

Proposition 1 (State-Redundancy) *If a differential game is perfect, then its open-loop Nash equilibrium is strongly time consistent (or subgame perfect).*

Proposition 2 (Uncontrollability) *If a differential game is uncontrollable by all of the players involved, then all of its open-loop Stackelberg equilibria are time consistent.*

Proposition 1 says that state-redundancy is necessary and sufficient to generate Markov-perfect open-loop Nash equilibria. However, note that the uncontrollability is necessary but not sufficient to generate Markov perfect equilibria in the open-loop Stackelberg game. The reason is that uncontrollability is unrelated to feedback effects, i.e., the presence of feedback effects throughout the game may prevent an uncontrollable game to yield strongly time consistent Stackelberg equilibria under the open-loop information structure. Together, Propositions 1-2 imply the following Corollary:

Corollary 3 *Consider an open-loop Stackelberg game. If it is both uncontrollable and perfect, then its Stackelberg equilibria are strongly time consistent (or subgame perfect). Otherwise, if it is perfect but controllable by at least one of the players, then the open-loop Stackelberg equilibrium with that player leading is bound to be time inconsistent.*

The above Corollary says that uncontrollability and state-redundancy must hold together in order for the open-loop Stackelberg behaviour to generate Markov-perfect solutions. More explicitly, in the Stackelberg case, state-redundancy is necessary but not sufficient to yield Markov-perfectness.

Observing (9) and (10), a further Remark emerges:

Remark 4 *The additive separability of each player's Hamiltonian function w.r.t. state and control variables is sufficient to ensure that the game is perfect and uncontrollable and all of its open-loop Nash and Stackelberg equilibria are strongly time consistent (or subgame perfect).*

The proof is trivial, in that additive separability implies that the first order condition of each player is independent of both the rival's control and state variables. However, it is worth stressing that additive separability is sufficient but by no means necessary to make the game perfect and uncontrollable, as it will become clear by examining the advertising game illustrated in the next section.[8]

[8] The analysis of dynamic oligopoly interaction has generated a relatively wide literature that cannot be exhaustively accounted for here. To the best of our knowledge, the earliest differential duopoly game can be found in Clemhout *et al.* (1971).

3 The CL advertising game

In this section, we explore a differential game of advertising where two firms, 1 and 2, invest in order to increase consumers' reservation prices for differentiated products, as in Cellini and Lambertini (2003; CL henceforth). Let p_i denote the price of good i, and q_i the quantity of good i. Firm i faces the following demand function, borrowed from Spence (1976) and also employed by Singh and Vives (1984) and Cellini and Lambertini (1998, 2002, 2004), *inter alia*:

$$p_i(t) = A_i(t) - Bq_i(t) - Dq_j(t) \tag{11}$$

Variable $A_i(t)$ describes the market size or the reservation price for good i. B and D are constant parameters, with $0 \leq D \leq B$. Notice that parameter D captures the degree of substitutability between any pair of different goods. In the limit case $D = 0$, goods are independent and each firm becomes a monopolist. In the opposite limit case $D = B$, the goods produced by different firms are perfect substitutes and the model collapses into the homogeneous oligopoly model.[9]

We assume that the market size may be increased by firms, through advertising activities. More precisely, we assume that the efforts made by any firm affects its own market size, as well as the market size of the rivals. In particular, the dynamics of the market size of firm i is described by the following equation:

$$\frac{dA_i(t)}{dt} \equiv \dot{A}_i(t) = k_i(t) + \gamma k_j(t) - \delta A_i(t) \tag{12}$$

where k_h is the effort in advertising made by firm h, $0 \leq \gamma \leq 1$ is a parameter capturing the external effect of the advertising of a firm on the market size of different firms, $\delta \geq 0$ is a constant depreciation parameter revealing that market size shrinks as time goes by.

[9] For a model where D is treated as a state variable and firms may invest so as to decrease product substitutability over time, see Cellini and Lambertini (2002, 2004).

Advertising activities are assumed to have decreasing marginal productivity, i.e., they entail a quadratic cost $\Gamma_i(k_i(t)) = \frac{\alpha}{2}(k_i(t))^2$ with $\alpha > 0$. Moreover, production entails a constant marginal production costs: $c(q_i(t)) = cq_i(t)$ with $c > 0$.

Quantities $\mathbf{q}(t)$ and advertising efforts $\mathbf{k}(t)$ are the control variables, while market sizes $\mathbf{A}(t)$ are the state variables. Each player chooses the path of her control variables over time, in order to maximize the present value of her profit flow, subject to the motion laws (12) of states, and initial conditions $\mathbf{A}(0) > c$, which are assumed to be known. Instantaneous profits are:

$$\pi_i(t) = \left(A_i(t) - Bq_i(t) - D\sum_{j \neq i} q_j(t) - c \right) q_i(t) - \frac{\alpha}{2}(k_i(t))^2 \tag{13}$$

so that player i's objective function is:

$$\max_{q_i(t), k_i(t)} J_i \equiv \int_0^\infty \pi_i(t) e^{-\rho t} dt \tag{14}$$

where the factor $e^{-\rho t}$ discounts future gains, and the discount rate ρ is assumed to be constant and common to all players.

To begin with, consider briefly the Nash game (for additional details, see CL, 2003, pp. 412-14). The Hamiltonian of firm i is:

$$\mathcal{H}_i(\mathbf{q}(t), \mathbf{k}(t), \mathbf{A}(t)) \equiv e^{-\rho t} \Big[(A_i(t) - Bq_i(t) - Dq_j(t) - c)\, q_i(t) - \frac{\alpha}{2}(k_i(t))^2 +$$

$$\lambda_{ii}(t) \cdot (k_i(t) + \gamma k_j(t) - \delta A_i(t)) + \lambda_{ij}(t) \cdot (k_j(t) + \gamma k_i(t) - \delta A_j(t)) \Big] \tag{15}$$

The above function is linear in the state variables $\mathbf{A}(t)$, so that the present game qualifies as a state linear one. As a consequence, its open-loop Nash equilibrium is strongly time consistent, and the game itself is perfect. Also note that the Hamiltonian function is additively separable in states and controls, a feature that, as anticipated in section 2, plays an obvious role in terms of the uncontrollability of the open-loop Stackelberg equilibrium.

3.1 The Stackelberg solution of the CL game

We stipulate that firm 1 is the leader and firm 2 is the follower. The follower's open-loop FOCs are:

$$\frac{\partial \mathcal{H}_2(\mathbf{q}(t), \mathbf{k}(t), \mathbf{A}(t))}{\partial q_2(t)} = A_2(t) - c - 2Bq_2(t) - Dq_1(t) = 0 \; ; \tag{16}$$

$$\frac{\partial \mathcal{H}_2(\mathbf{q}(t), \mathbf{k}(t), \mathbf{A}(t))}{\partial k_2(t)} = -\alpha k_2(t) + \lambda_{22}(t) + \gamma \lambda_{21}(t) = 0 \; ; \tag{17}$$

$$-\frac{\partial \mathcal{H}_2(\mathbf{q}(t), \mathbf{k}(t), \mathbf{A}(t))}{\partial A_2(t)} = \frac{\partial \lambda_{22}(t)}{\partial t} - \rho \lambda_{22} \Leftrightarrow \tag{18}$$

$$\frac{\partial \lambda_{22}(t)}{\partial t} = -q_2(t) + \lambda_{22}(t)(\delta + \rho)$$

$$-\frac{\partial \mathcal{H}_2(\mathbf{q}(t), \mathbf{k}(t), \mathbf{A}(t))}{\partial A_1(t)} = \frac{\partial \lambda_{21}(t)}{\partial t} - \rho \lambda_{21} \Leftrightarrow \tag{19}$$

$$\frac{\partial \lambda_{21}(t)}{\partial t} = \lambda_{21}(t)(\delta + \rho)$$

These conditions have to be considered together with the initial conditions $\mathbf{A}(0) = \mathbf{A}_0$ and the transversality conditions:

$$\lim_{t \to \infty} e^{-\rho t} \cdot \lambda_{ij}(t) \cdot A_j(t) = 0, \; i, \, j = 1, 2; \; j \neq i. \tag{20}$$

Now, note that (19) is a separable differential equation admitting the solution $\lambda_{21}(t) = 0$ at all t. Using this result, (17) becomes:

$$\frac{\partial \mathcal{H}_2(\mathbf{q}(t), \mathbf{k}(t), \mathbf{A}(t))}{\partial k_2(t)} = -\alpha k_2(t) + \lambda_{22}(t) = 0, \tag{21}$$

so that $\lambda_{22}(t) = \alpha k_2(t)$, which is independent of the leader's controls. Moreover, from (18) we have that the dynamics of $\lambda_{22}(t)$ is also independent of the leader's controls. Therefore, we may state the following:

Proposition 5 *The CL advertising game is uncontrollable by the leader. Since the game is also perfect, than all of its open-loop equilibria are strongly time consistent irrespective of the timing of moves.*

In the next section, we explore an alternative advertising game where the mark-up is exogenously given and advertising is aimed at increasing market shares arther than consumers' reservation prices.

4 The LS advertising game

As in Leitmann and Schmitendorf (1978; LS henceforth) and Feichtinger (1983), we have a non cooperative differential game over $t \in [0, T]$ between two firms, 1 and 2, choosing their respective advertising efforts $u_i(t)$ to maximise their individual discounted payoff:

$$J_i = \int_0^T [p_i x_i(t) - u_i(t)] e^{-\rho t} dt, \; i = 1, 2 \tag{22}$$

where $x_i(t)$, firm i's market share, is a state variable evolving according to:

$$\dot{x}_i = -\beta_i x_i(t) + u_i(t) - \frac{1}{2} c_i u_i^2(t) - k_i x_i(t) u_j(t), \; i, \; j = 1, 2; \; j \neq i. \tag{23}$$

p_i, β_i, c_i and k_i are positive parameters. In particular, $\beta_i \in [0, 1]$ is the decay rate of firm i's market share,[10] while k_i measures the spillover from the rival's investment $u_j(t)$ in proportion to firm i's current demand $x_i(t)$. The factor $e^{-\rho t}$ discounts future gains, and the discount rate ρ is assumed to be constant and common to all players.[11] In addition to (23), we also adopt the further constraint whereby $u_i(t) \in [0, 1/c_i]$, which amounts to saying that there is an upper bound to the advertising investment of firm i, $i = 1, 2$. This restriction remains to be checked ex post, once we are in a position to determine the features of the steady state equilibrium.

The Hamiltonian of firm i is:

$$\mathcal{H}_i(\mathbf{x}(t), \mathbf{u}(t)) = e^{-\rho t} \left\{ [p_i x_i(t) - u_i(t)] + \lambda_{ii}(t) \dot{x}_i(t) + \lambda_{ij}(t) \dot{x}_j(t) \right\} \tag{24}$$

with i, $j = 1, 2$; $j \neq i$, where the scrap value at the terminal date T is set equal to zero for the sake of simplicity, and without further loss of generality. We know from LS that the Nash open-loop solution is a degenerate

[10] To ensure that $x_i(t) + x_j(t) \leq 1$, one has to impose $\beta_i \geq 1/(2c_i) + 1/(2c_j)$ (Lemma 1 in LS, p. 646).

[11] In the original version of the LS model, the discount rate is nil. As we show in the remainder, introducing positive discounting does not modify significantly the conclusions.

closed-loop one, i.e., there exists a state-independent feedback control for each player. Using

$$\dot{\lambda}_{ij}(t) = \lambda_{ij}(t)\left[k_j u_i^*(t) + \beta_j + k_j x_j^*(t)\frac{\partial u_i^*(t)}{\partial x_j} + \rho\right] + \\ -\lambda_{ii}(t)\left[1 - c_i u_i^*(t)\right]\frac{\partial u_i^*(t)}{\partial x_j} = 0 \tag{25}$$

$$\frac{\partial u_i^*(t)}{\partial x_j(t)} = -\frac{\lambda_{ij}(t)\,k_j}{\lambda_{ii}(t)\,c_i},\ i,\ j = 1,2;\ j \neq i \tag{26}$$

one finds that (25) is a separable differential equation which admits the solution $\lambda_{ij}(t) = 0$ for $j \neq i$, at all t. This, in turn, entails that the first order condition:[12]

$$\frac{\partial \mathcal{H}_i(\mathbf{x}^*(t), \mathbf{u}^*(t))}{\partial u_i} = 0 \Leftrightarrow u_i^*(t) = \frac{\lambda_{ii}(t) - 1 - \lambda_{ij} k_j x_j^*(t)}{c_i \lambda_{ii}(t)} \tag{27}$$

can be rewritten as follows:

$$\begin{aligned} u_i^*(t) &= \frac{\lambda_{ii}(t) - 1}{c_i \lambda_{ii}(t)} \geq 0 \text{ iff } \lambda_{ii}(t) \geq 1; \\ u_i^*(t) &= 0 \text{ otherwise; } i = 1,2 \end{aligned} \tag{28}$$

At this point, from (28) there emerges the property $\partial u_i^*(t)/\partial x_j(t) = 0$, so that the open-loop solution is a degenerate closed-loop one, and yields a Markov equilibrium. The aforementioned property is equivalent to (10). We can summarise the above discussion in the following:

Lemma 6 (Leitmann and Schmitendorf, 1978) *The open-loop Nash solution of the LS game is strongly time consistent. Therefore, the LS game is perfect.*

It is worth stressing that the above method is that employed by Mehlmann (1988) in revisiting the LS model. The alternative method adopted by Leitmann and Schmitendorf consists in showing that the open-loop controls depend only upon time but not states and initial conditions. In both cases,

[12]Global sufficiency conditions, which are omitted for brevity, can be shown to hold (Stalford and Leitmann, 1973).

one ascertains that the feedback effects at any instant t during the game are endogenously nil, and therefore the open-loop Nash equilibrium is indeed Markovian, i.e., it is the collapse of a closed-loop equilibrium.

To complete the characterisation of the open-loop Markovian equilibrium, from (28) we obtain:[13]

$$\dot{u}_i^* = \frac{\dot{\lambda}_{ii}}{c_i \lambda_{ii}^2} \tag{29}$$

provided $\lambda_{ii} \geq 1$. The dynamics of λ_{ii} is given by the following co-state equation, derived under the open-loop information structure:

$$-\frac{\partial \mathcal{H}_i (\mathbf{x}^*, \mathbf{u}^*)}{\partial x_i} = \dot{\lambda}_{ii} - \rho \lambda_{ii} \Leftrightarrow \dot{\lambda}_{ii} = \lambda_{ii} \left(\rho + \beta_i + k_i u_j^* \right) - p_i \, . \tag{30}$$

The transversality condition is $\lambda_{ij} (T) = 0$, i, $j = 1, 2$.

Expression (28) can be rearranged to obtain $\lambda_{ii} = 1/ (1 - c_i u_i^*)$ which, together with (30) can be substituted into (29) to obtain:

$$\dot{u}_i^* = \frac{(1 - c_i u_i^*) \left[\rho + \beta_i - p_i (1 - c_i u_i^*) + k_j u_i^* \right]}{c_i}, \; i, \; j = 1, 2; \; j \neq i. \tag{31}$$

The steady state values of optimal controls and states can be found by solving the system $\left(\dot{x}_i = 0; \dot{u}_i^* = 0 \right)$ w.r.t. $\mathbf{x}$ and $\mathbf{u}$. To simplify the exposition and focus the attention on the fundamental properties of our analysis, at this point we may introduce some symmetry conditions, whereby $c_i = c_j = c$, $k_i = k_j = k$, $p_i = p_j = p$, $\beta_i = \beta_j = \beta$, $u_i = u_j = u$ and $x_i = x_j = x$. In this way, we confine to symmetric steady state equilibria.[14] The stationarity condition $\dot{x} = 0$ yields $x^s = u^* (2 - cu^*) / [2 (\beta + ku^*)]$ which can be plugged into (31) so that $\dot{u}^* = 0$ in $u_a \equiv 1/c$; $u_b \equiv (p - \beta - \rho) / (cp + k)$. Note that $u_b < u_a$ always, and $u_b > 0$ iff $p > \beta + \rho$. The latter condition means that, in order for the steady state advertising effort to be positive, the marginal

[13] The indication of time and discounting is omitted henceforth, whenever inessential.

[14] It can also be shown that indeed a unique steady state equilibrium exists, and it is symmetric, even without imposing the above symmetry conditions.

value that the firm attaches to any additional customer (p) must be higher than the sum of depreciation and discount rates.[15]

We may summarise the foregoing analysis in:

Proposition 7 *The open-loop Nash game yields a unique saddle point equilibrium where the symmetric steady state values of the Markovian controls are* $u^s = (p - \beta - \rho) / (cp + k)$. *The corresponding steady state level of the state variable is:*

$$x^s = \frac{(p - \beta - \rho) [2k + c (p + \beta + \rho)]}{2 (c\beta + k) [cp\beta + k (p - \rho)]}.$$

Now we move on to the Stackelberg open-loop solution of the same game, which can be shown to be subgame (Markov) perfect as well.

4.1 The Stackelberg solution of the LS game

To simplify exposition, as in the CL game we stipulate that firm 1 is the leader and firm 2 is the follower. The follower maximizes

$$\mathcal{H}_2 (\mathbf{x}(t), \mathbf{u}(t)) = e^{-\rho t} \left\{ [p_2 x_2 (t) - u_2 (t)] + \lambda_{22} (t) \dot{x}_2 (t) + \lambda_{21} (t) \dot{x}_1 (t) \right\} \tag{32}$$

where

$$\begin{aligned} \dot{x}_1 (t) &= -\beta_1 x_1 (t) + u_1 (t) - \tfrac{1}{2} c_1 u_1^2 (t) - k_1 x_1 (t) u_2 (t) ; \\ \dot{x}_2 (t) &= -\beta_2 x_2 (t) + u_2 (t) - \tfrac{1}{2} c_2 u_2^2 (t) - k_2 x_2 (t) u_1 (t) . \end{aligned} \tag{33}$$

The FOCs for the open-loop solution are:

$$\frac{\partial \mathcal{H}_2 (\mathbf{x}^*, \mathbf{u}^*)}{\partial u_2} = \lambda_{22} [1 - c_2 u_2^*] - 1 - \lambda_{21} k_1 x_1^* = 0 \Leftrightarrow \tag{34}$$

$$u_2^* = \frac{\lambda_{22} - 1 - \lambda_{21} k_1 x_1^*}{c_2 \lambda_{22}} \tag{35}$$

[15] The stability analysis is omitted for brevity. However, it can be shown that the positivity of the advertising investment is sufficient to ensure that (x^s, u_b) is a saddle point. For further details on this aspect, see Cellini *et al.* (2004).

$$\frac{\partial \mathcal{H}_2(\mathbf{x}^*,\mathbf{u}^*)}{\partial x_2} = \dot{\lambda}_{22} - \rho\lambda_{22} \Leftrightarrow \dot{\lambda}_{22} = \lambda_{22}(k_2 u_1^* + \beta_2 + \rho) - p_2 \tag{36}$$

$$\frac{\partial \mathcal{H}_2(\mathbf{x}^*,\mathbf{u}^*)}{\partial x_1} = \dot{\lambda}_{21} - \rho\lambda_{21} \Leftrightarrow \dot{\lambda}_{21} = \lambda_{21}(k_1 u_2^* + \beta_1 + \rho) \tag{37}$$

From (34), observe that $\partial\lambda_{22}/\partial u_1 = \partial^2\mathcal{H}_2(\mathbf{x}^*,\mathbf{u}^*)/\partial u_2\partial u_1 = 0$ which entails the following result:

Lemma 8 *The Stackelberg LS game is uncontrollable by the leader. Therefore, the open-loop Stackelberg solution is time consistent.*

Lemmas 5 and 8 imply:

Proposition 9 *The LS model is a perfect uncontrollable game. Therefore, all of its Nash and Stackelberg open-loop equilibria are subgame perfect.*

The explicit characterisation of the open-loop Stackelberg equilibrium can be found in Cellini *et al.* (2004).

5 Extensions and concluding remarks

We have analysed the time consistency property of open-loop equilibria, in the case of Nash and Stackelberg behaviour. We have noted that classes of games exist, in which the strong time-consistency of the open-loop Nash equilibrium associates with the time consistency of the open-loop Stackelberg equilibrium. We have labelled these setups as *perfect uncontrollable games*. We have also provided two examples based on different models of oligopolistic competition with advertising efforts analysed by Cellini and Lambertini (2003) and Leitmann and Schmitendorf (1978), respectively.

We have confined our attention to two-player games, and the generalisation to the case of N players is desirable. In addition to that, investigating whether the class of linear state games enjoys the property of being *always* perfect *and* uncontrollable is also an interesting task. These extensions are left for future research.

References

[1] Basar, T., and Olsder, G.J. (1982, 1995^2), *Dynamic Noncooperative Game Theory*, San Diego, Academic Press.

[2] Cellini, R. and L. Lambertini (1998), "A Dynamic Model of Differentiated Oligopoly with Capital Accumulation", *Journal of Economic Theory*, **83**, 145-55.

[3] Cellini, R., and L. Lambertini (2002), "A Differential Game Approach to Investment in Product Differentiation", *Journal of Economic Dynamics and Control*, **27**, 51-62.

[4] Cellini, R. and L. Lambertini (2003), "Advertising with Spillover Effects in a Differential Oligopoly Game with Differentiated Goods", *Central European Journal of Operations Research*, **11**, 409-23.

[5] Cellini, R. and L. Lambertini (2004), "Private and Social Incentives Towards Investment in Product Differentiation", *International Game Theory Review*, **6**, 493-508.

[6] Cellini, R., L. Lambertini and G. Leitmann (2004), "Perfect Uncontrollable Differential Games", wp 509, Dipartimento di Scienze Economiche, Università degli Studi di Bologna.

[7] Clemhout, S., G. Leitmann and H.Y. Wan, Jr. (1971), "A Differential Game Model of Duopoly", *Econometrica*, **39**, 911-38.

[8] Clemhout, S. and H.Y. Wan, Jr. (1974), "A Class of Trilinear Differential Games", *Journal of Optimization Theory and Applications*, **14**, 419-24.

[9] Dockner, E.J, S. Jørgensen, N. Van Long and G. Sorger (2000), *Differential Games in Economics and Management Science*, Cambridge, Cambridge University Press.

[10] Feichtinger, G. (1983), "The Nash Solution of an Advertising Differential Game: Generalization of a Model by Leitmann and Schmitendorf", *IEEE Transactions on Automatic Control*, **28**, 1044-48.

[11] Fershtman, C. (1987), "Identification of Classes of Differential Games for Which the Open-Loop is a Degenerated Feedback Nash Equilibrium", *Journal of Optimization Theory and Applications*, **55**, 217-31.

[12] Fershtman, C., M. Kamien and E. Muller (1992), "Integral Games: Theory and Applications", in Feichtinger, G. (ed.), *Dynamic Economic Models and Optimal Control*, Amsterdam, North-Holland, 297-311.

[13] Leitmann, G. and W.E. Schmitendorf (1978), "Profit Maximization through Advertising: A Nonzero Sum Differential Game Approach", *IEEE Transactions on Automatic Control*, **23**, 646-50.

[14] Mehlmann, A. (1988), *Applied Differential Games*, New York, Plenum Press.

[15] Mehlmann, A. and R. Willing (1983), "On Nonunique Closed-Loop Nash Equilibria for a Class of Differential Games with a Unique and Degenerate Feedback Solution", *Journal of Optimization Theory and Applications*, **41**, 463-72.

[16] Persson, T. and G. Tabellini (2000), *Political Economics. Explaining Economic Policy*, Cambridge, MA, MIT Press.

[17] Reinganum, J. (1982), "A Class of Differential Games for Which the Closed Loop and Open Loop Nash Equilibria Coincide", *Journal of Optimization Theory and Applications*, **36**, 253-62.

[18] Simaan, M. and J.B. Cruz, Jr. (1973a), "On the Stackelberg Strategy in Nonzero Sum Games", *Journal of Optimization Theory and Applications*, **11**, 533-55.

[19] Simaan, M. and J.B. Cruz, Jr. (1973b), "Additional Aspects of the Stackelberg Strategy in Nonzero Sum Games", *Journal of Optimization Theory and Applications*, **11**, 613-26.

[20] Singh, N. and X. Vives (1984), "Price and Quantity Competition in a Differentiated Duopoly", *RAND Journal of Economics*, **15**, 546-54.

[21] Spence, A.M. (1976), "Product Differentiation and Welfare", *American Economic Review*, **66**, 407-14.

[22] Stalford, H. and G. Leitmann (1973), "Sufficiency Conditions for Nash Equilibria in N-Person Differential Games", in Blaquiere, A. (ed.), *Topics in Differential Games*, Amsterdam, Elsevier.

[23] Xie, D. (1997), "On Time Consistency: A Technical Issue in Stackelberg Differential Games", *Journal of Economic Theory*, **76**, 412-30.