Identifying Reaction Functions in a Differential Oligopoly Game with Sticky Prices[1]

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Abstract

We investigate the issue of strategic substitutability/complementarity in a Cournot differential game with sticky prices. We show that first order conditions do not produce instantaneous best reply functions. However, we identify negatively sloped reaction functions in steady state, with the open-loop best reply being flatter than its closed-loop counterpart.

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1 Introduction

The issue of super-/submodularity has been investigated mostly in static games, and refers to the slope of reaction functions in the (stage) game, as initially pointed out by Bulow, Geanakoplos and Klemperer (1985).[1] A potential development of this discussion consists in investigating whether the same properties can be reconstructed in a differential game, and to what extent.[2]

To the best of our knowledge, only Jun and Vives (2004) have considered intertemporal strategic complementarity/substitutability. They compare steady states of open-loop and stable closed-loop equilibria in a general symmetric differential duopoly model with adjustment costs, as in Reynolds (1987) and Driskill and McCafferty (1989). One of the most interesting result appears in the "mixed" case of price competition and production adjustment costs: the strategic complementarity of the static game turns into an intertemporal strategic substitutability, given that, from the standpoint of any given firm, a price cut today makes the rival smaller in the future by raising its short-run marginal cost.

[1]The concept of economic complementarity has recently emerged as a leading theme of economic research and has benefitted from the development of the theory of supermodular games, introduced by Topkis (1978) and based on lattice-theoretic arguments. The analysis has been focused on games with strategic complementarities and their use in industrial economics (Vives, 1990; Milgrom and Roberts, 1990 and Amir, 1996) and in comparative statics analysis (Milgrom and Shannon, 1994). In the presence of complementarity relationships between different units of a global system, a separate study of any unit alone, *ceteris paribus*, could lead to a wrong interpretation of the phenomenon under consideration.

[2]See Dockner *et al.* (2000) and Mehlmann (1988) for the theory of differential games and its appications to economics.

A related issue is that of conjectural variations. Dockner (1992) shows that any closed-loop (i.e., subgame perfect) equilibrium coincides with a conjectural variations equilibrium. Using a Cournot model with linear demand and quadratic production costs, he proves that the dynamic conjectural variations consistent with the closed-loop equilibrium are negative constants which, depending upon the level of the discount rate, vary between zero and the consistent conjectures characterising the static version of the same game.

The aim of the present paper is to identify best reply functions in a Cournot differential game with sticky prices *à la* Simaan and Takayama (1978) and Fershtman and Kamien (1987). In general, a dynamic game of the type proposed here may either generate instantaneous best replies directly from the first order conditions on controls, or yield best replies at the steady state only. The emergence of the first or the second case ultimately depends upon whether the first order condition taken w.r.t. the output level of any given firm contains the outputs of her rivals or not. In the model we investigate, the second case holds both under the open-loop solution and the closed-loop one. This implies that, at any time during the game, each firm has a dominant strategy independent of the rivals' behaviour. A proper strategic interaction only emerges when one imposes stationarity on the dynamics of state and control variables. At the steady state, best replies are negatively sloped, with closed-loop best replies being always steeper than open-loop ones, to indicate that strategic interaction is stronger in the former case than in the latter. Moreover, we also show that, if price stickiness is infinitely high, the types of equilibria coincide with the perfectly competitive outcome which can be computed in the static model.

The plan of the paper is as follows. Section 2 introduces the issue of identifying reaction functions using a general differential game framework. The sticky price game, and its open-loop and closed-loop solutions are investigated in section 3. Section 4 concludes.

2 Preliminaries

Consider a generic differential game, played over continuous time, with $t \in [0, \infty)$.[3] The set of players is $\mathbb{P} \equiv \{1, 2, ...N\}$. Moreover, let $x_i(t)$ and $u_i(t)$ define, as usual, the state variable and the control variable pertaining to player i. Assume there exists a prescribed set $\mathcal{U}_i$ such that any admissible action $u_i(t) \in \mathcal{U}_i$. The dynamics of player i's state variable is described by the following:

$$\frac{dx_i(t)}{dt} \equiv \dot{x}_i(t) = f_i\left(\mathbf{x}(t), \mathbf{u}(t)\right) \tag{1}$$

where $\mathbf{x}(t) = \left(x_1\left(t\right), x_2\left(t\right), ...x_N\left(t\right)\right)$ is the vector of state variables at time t, and $\mathbf{u}(t) = \left(u_1\left(t\right), u_2\left(t\right), ...u_N\left(t\right)\right)$ is the vector of players' actions at the same date, i.e., it is the vector of control variables at time t. That is, in the most general case, the dynamics of the state variable associated with player i depends on all state and control variables associated with all players involved in the game. The value of the state variables at $t = 0$ is assumed to be known: $\mathbf{x}(0) = \left(x_1\left(0\right), x_2\left(0\right), ...x_N\left(0\right)\right)$.

Each player has an objective function, defined as the discounted value of the flow of payoffs over time. The instantaneous payoff depends upon the choices made by player i as well as its rival, that is:

$$\pi_i \equiv \pi_i\left(\mathbf{x}(t), \mathbf{u}(t)\right). \tag{2}$$

[3] One could also consider a finite terminal time T. The specific choice of the time horizon is immaterial to the ensuing analysis, provided that terminal conditions are appropriately defined.

Player i's objective is then, given $u_j(t), j \neq i$:

$$\max_{u_i(\cdot)} J_i \equiv \int_0^\infty \pi_i(\mathbf{x}(t), \mathbf{u}(t)) e^{-\rho t} dt \tag{3}$$

subject to the dynamic constraint represented by the behaviour of the state variables, (1), $u_i(t) \in \mathcal{U}_i$ and initial conditions $\mathbf{x}(0) = (x_1(0), x_2(0), ...x_N(0))$.

The Hamiltonian of player i is:

$$\begin{aligned} \mathcal{H}_i(\mathbf{x}(t), \mathbf{u}(t)) \equiv\ & e^{-\rho t}[\pi_i(\mathbf{x}(t), \mathbf{u}(t)) + \lambda_{ii}(t) \cdot f_i(\mathbf{x}(t), \mathbf{u}(t)) + \\ & + \sum_{j \neq i} \lambda_{ij}(t) \cdot f_j(\mathbf{x}(t), \mathbf{u}(t))] , \end{aligned} \tag{4}$$

where $\lambda_{ij}(t) = \mu_{ij}(t)e^{\rho t}$ is the co-state variable (evaluated at time t) that firm i associates with the state variable $x_j(t)$.

The interesting property, in the present perspective, is summarised by the second cross-derivative w.r.t. controls:

$$\frac{\partial^2 \mathcal{H}_i}{\partial u_i \partial u_j} \tag{5}$$

Two cases are possible:

- If

$$\frac{\partial^2 \mathcal{H}_i}{\partial u_i \partial u_j} \neq 0 \tag{6}$$

  then the first order condition (FOC)

$$\frac{\partial \mathcal{H}_i}{\partial u_i} = 0 \tag{7}$$

  yields the instantaneous best reply function of player i against any admissible choice of player j at any time t, and

$$sgn\left(\frac{\partial^2 \mathcal{H}_i}{\partial u_i \partial u_j}\right) \tag{8}$$

  is the slope of such best reply function.

- If instead
$$\frac{\partial^2 \mathcal{H}_i}{\partial u_i \partial u_j} = 0\, \forall j \neq i, \tag{9}$$
then the FOC does not yield an instantaneous best reply function. The necessary and sufficient condition for (9) to hold is additive separability of the Hamiltonian of player i w.r.t. control variables. In this situation, it must be nonetheless true that the expression $\partial \mathcal{H}_i / \partial u_i$ contains the co-state variables. Hence, in order to solve for the equilibrium path of u_i, one has to take the derivative of (7) w.r.t. t. This yields:
$$\dot{u}_i = z_i \left(\dot{\lambda}_{ii}, \dot{\lambda}_{ij}, \dot{x}_i, \dot{x}_j \right) \tag{10}$$
where $\dot{x}_i, \dot{x}_j$ are given by state equations (1), and the dynamics of the co-state variables λ_{ii} and λ_{ij} comes from the co-state equations:
$$-\frac{\partial \mathcal{H}_i}{\partial x_i} - \sum_{j \neq i} \frac{\partial \mathcal{H}_i}{\partial u_j} \frac{\partial u_j^*}{\partial x_i} = \dot{\lambda}_{ii} - \rho \lambda_{ii} \tag{11}$$
If (11) contains (u_i, u_j), then, by substitution, we will observe $\dot{u}_i = w_i (u_i, u_j)$. Imposing $\dot{u}_i = 0$, one obtains $u_i^* = v_i (u_j)$ representing the best reply against the choice of j in the steady state equilibrium.

The difference between the two cases lies in the fact that while in the first case we observe an instantaneous reaction function characterising the optimal behaviour of player i at any time during the game, in the second case we only observe player i's best reply at the steady state equilibrium, while i's optimal behaviour during the transition to the steady state can be characterised in terms of states and co-states only, regardless of any player j's control. This amounts indeed to saying that along the path to the steady state each player has a dominant strategy. This discussion is summarised by:

Remark 1 *If player i's Hamiltonian is additively separable w.r.t. controls, then* $\partial^2 \mathcal{H}_i / \partial u_i \partial u_j = 0$ *and each player i has a dominant strategy at every instant.*

To better illustrate this point, we resort to a model with sticky prices *à la* Fershtman and Kamien (1987).

3 Sticky prices

This model dates back to Simaan and Takayama (1978) and Fershtman and Kamien (1987). Consider an oligopoly where, at any $t \in [0, \infty)$, N single-product firms produce quantities $q_i(t)$, $i \in \{1, 2, ...N\}$, of the same homogeneous good at a total cost $C_i(t) = cq_i(t) + [q_i(t)]^2/2$, $c > 0$. In each period, market demand determines the notional price level:

$$\widehat{p}(t) = A - \sum_{i=1}^{N} q_i(t).$$

In general, however, $\widehat{p}(t)$ will differ from the current price level $p(t)$, due to price stickiness, and price moves according to the following equation:

$$\frac{dp(t)}{dt} \equiv \dot{p} = s \{\widehat{p}(t) - p(t)\} \tag{12}$$

Notice that the dynamics described by (12) establishes that price adjusts proportionately to the difference between the price level given by the inverse demand function and the current price level, the speed of adjustment being determined by the constant $s \in [0, \infty)$.

This amounts to saying that the price mechanism is sticky, that is, firms face menu costs in adjusting their price to the demand conditions deriving from consumers' preferences: they may not (and, in general, they will not) choose outputs so that the price reaches immediately $\widehat{p}(t)$, except in the limit case where s tends to infinity.

The instantaneous profit function of firm i is:

$$\pi_i(t) = q_i(t) \cdot \left[p(t) - c - \frac{1}{2}q_i(t)\right] . \tag{13}$$

Hence, the problem of firm i is:

$$\max_{q_i(t)} J_i = \int_0^\infty e^{-\rho t} \, q_i(t) \cdot \left[p(t) - c - \frac{1}{2}q_i(t)\right] \, dt \tag{14}$$

subject to (12) and to the conditions $p(0) = p_0$, and $p(t) \geq 0$ for all $t \in [0, \infty)$.

We solve the game by considering - in turn - the open-loop solution and the closed-loop memoryless solution.[4]

3.1 The open-loop solution

Here we look for the open-loop Nash equilibrium, i.e., we examine a situation where firms commit to a production plan at $t = 0$ and stick to that plan forever.

The Hamiltonian function is:

$$\mathcal{H}_i(t) = e^{-\rho t} \cdot \left\{ q_i(t) \cdot \left[p(t) - c - \frac{1}{2}q_i(t)\right] + \lambda_i(t) s \left[A - \sum_{i=1}^{N} q_i(t) - p(t)\right]\right\}, \tag{15}$$

where $\lambda_i(t) = \mu_i(t)e^{\rho t}$, and $\mu_i(t)$ is the co-state variable associated to $p(t)$. In the remainder of the section, superscript OL indicates the *open-loop* equilibrium level of a variable. Consider the first order condition (FOC) w.r.t.

[4]For a thourough analysis of open-loop, memoryless closed-loop and fedback solutions, see Cellini and Lambertini (2004).

$q_i(t)$, calculated using (15):

$$\frac{\partial \mathcal{H}_i(t)}{\partial q_i(t)} = p(t) - c - q_i(t) - \lambda_i(t)s = 0\,. \tag{16}$$

This yields the optimal open-loop output for firm i, as follows:[5]

$$q_i(t) = \begin{cases} p(t) - c - \lambda_i(t)s & \text{if } p(t) > c + \lambda_i(t)s \\ 0 & \text{otherwise.} \end{cases} \tag{17}$$

Observe that, on the basis of (16-17), one cannot write a best reply function for player i against his rivals at time t.

The adjoint condition for the optimum are:

$$-\frac{\partial \mathcal{H}_i(t)}{\partial p(t)} = \dot{\mu}_i \Leftrightarrow \dot{\lambda}_i = \lambda_i(t)(s+\rho) - q_i(t)\,, \tag{18}$$

while the transversality condition requires:

$$\lim_{t\to\infty} \mu_i(t)\cdot p(t) = 0\,. \tag{19}$$

Differentiating (17), we obtain:

$$\dot{q}_i = \dot{p} - s\dot{\lambda}_i \tag{20}$$

which, using (18), can be rewritten as follows:

$$\dot{q}_i = \dot{p} - s\left[(\rho+s)\lambda_i(t) - q_i(t)\right]. \tag{21}$$

Now, substitute into (21) (i) the law of motion of the price, $\dot{p} = s\left\{\widehat{p}(t) - p(t)\right\}$, with $\widehat{p}(t) = A - q_i(t) - \sum_{j\neq i} q_j(t)$, and (ii) $s\lambda_i(t) = p(t) - c - q_i(t)$ from (17). This yields:

$$\dot{q}_i = s\left[A - q_i(t) - \sum_{j\neq i} q_j(t) - p(t)\right] - (s+\rho)\left[p(t) - c - q_i(t)\right] + sq_i(t) = 0 \tag{22}$$

[5] In the remainder, we consider the positive solution. Obviously, the derivation of the steady state entails non-negativity constraints on price and quantity, that we assume to be satisfied.

in:

$$q_i^*(t) = \frac{s\sum_{j\neq i} q_j\,(t) - sA - (s+\rho)c + (2s+\rho)p(t)}{s+\rho}. \qquad (23)$$

Now imposing stationarity on (12), we obtain:

$$\dot{p} = 0 \text{ in } p^* = \widehat{p} = A - q_i(t) - \sum_{j\neq i} q_j\,(t) \qquad (24)$$

which can be plugged into (23), yielding:

$$q_i^* = \frac{(s+\rho)\left(A - c - \sum_{j\neq i} q_j\right)}{3s+2\rho}. \qquad (25)$$

Expression (25) is the best reply function of firm i at the open-loop equilibrium, with slope:

$$\frac{\partial q_i^*}{\partial q_j} = -\frac{s+\rho}{3s+2\rho} < 0\,. \qquad (26)$$

This allows us to state the following:

Lemma 2 *At the open-loop equilibrium, the best reply function of firm i is negatively sloped for all $s, \rho \in [0,\infty)$. In absolute value, the slope is everywhere decreasing in s. In the limit, as s tends to infinity (or ρ tends to zero), $\partial q_i^*/\partial q_j = -1/3$, while as s tends to zero (or ρ tends to infinity), $\partial q_i^*/\partial q_j = -1/2$.*

Now we can introduce the symmetry condition $q_j(t) = q_i(t)$ for all j, so as to obtain:

$$\frac{dq(t)}{dt} = sA + (s+\rho)c - (2s+\rho)p(t) + [\rho - Ns]\,q(t) \qquad (27)$$

Note that $dq(t)/dt = 0$ is a linear relationship between $p(t)$ and $q(t)$. This, together with $dp(t)/dt = 0$, also a linear function, fully characterise the steady state of the system. The dynamic system can be immediately rewritten in matrix form as follows:

$$\begin{bmatrix} \dot{p} \\ \\ \dot{q} \end{bmatrix} = \begin{bmatrix} -s & -sN \\ -(2s+\rho) & \rho - sN \end{bmatrix} \begin{bmatrix} p \\ \\ q \end{bmatrix} + \begin{bmatrix} sA \\ \\ sA + (s+\rho)c \end{bmatrix} \tag{28}$$

As the determinant of the above 2×2 matrix is negative, the equilibrium point is a saddle, with

$$q^{OL} = \frac{(A-c)(s+\rho)}{(s+\rho)(N+1)+s} \; ; \; p^{OL} = A - Nq^{OL} \, . \tag{29}$$

As in the duopoly case described by Fershtman and Kamien (1987 pp. 1159-61), also here the static Cournot-Nash equilibrium price and output $\left\{p^{CN}, q^{CN}\right\}$ obtain from (29), in the limit, when $\rho \to 0$ or $s \to \infty$. For all positive levels of the discount rate and for any finite speed of adjustment, the static Cournot price (output) is higher (lower) than the open-loop equilibrium price (output). Moreover, for either $\rho \to \infty$ or $s \to 0$, the open-loop steady state equilibrium $\left\{p^{CN}, q^{CN}\right\}$ exactly replicates the perfectly competitive outcome that would emerge from the static model, for any $N \geq 1$ (see Fershtman and Kamien, 1987, Proposition 1, p. 1156):

$$\lim_{s \to 0} q^{OL} = \lim_{\rho \to \infty} q^{OL} = \frac{A-c}{N+1} \; ; \; \lim_{s \to 0} p^{OL} = \lim_{\rho \to \infty} p^{OL} = \frac{A+Nc}{N+1} \, . \tag{30}$$

From Lemma 2, we know that in such a case $\partial q_i^* / \partial q_j = -1/2$; hence, one may ask why the price coincides with marginal cost in a game with negatively sloped best replies, and, even more striking, in the monopoly setting which obtains when $N = 1$. The intuitive explanation is that, if $s = 0$, then firms are unable to affect the notional price through any change in output levels, which is the only available instrument to influence the variation of the current price level. Therefore, independently of market structure, when the current price is infinitely sticky, firms are obliged to behave as price-takers, as if they were supplying a perfectly competitive market. This of course holds for a monopolist as well. In the static game, it can be easily ascertained that the perfectly competitive outcome is reached only when N tends to infinity.

Finally, if either s tends to infinity or ρ tends to zero, the open-loop steady state equilibrium coincides with the static Cournot-Nash equilibrium where $q^{CN} = (A - c) / (N + 2)$, as it can be verified using q^{OL} in (29).

3.2 The closed-loop solution

The closed-loop memoryless solution remains to investigate. We use superscript CL to denote the *closed-loop* equilibrium levels of the relevant variables. The Hamiltonian of firm i is given by (15), with the same initial and transversality conditions. The first order condition w.r.t. q_i, calculated using (15), obviously coincide with condition (16) calculated in the open-loop case:

$$\frac{\partial \mathcal{H}_i}{\partial q_i} = p - c - q_i - \lambda_i s = 0\,. \tag{31}$$

This yields the closed-loop output for firm i, as follows (again, in the remainder we shall consider only the positive solution):

$$q_i^{CL} = \begin{cases} p - c - \lambda_i s & \text{if } p > c + \lambda_i s \\ 0 & \text{otherwise.} \end{cases} \tag{32}$$

Note that the kinematic equation of q_i^{CL} is described by (20), as in the open-loop case. The adjoint conditions for the optimum are:

$$-\frac{\partial \mathcal{H}_i}{\partial p} - \sum_{j \neq i} \frac{\partial \mathcal{H}_i}{\partial q_j} \frac{\partial q_j^{CL}}{\partial p} = \dot{\lambda}_i - \rho \lambda_i \tag{33}$$

Now consider that

$$\frac{\partial \mathcal{H}_i}{\partial q_j} = -\lambda_i s \;;\; \frac{\partial q_j^{CL}}{\partial p} = 1\,. \tag{34}$$

Therefore:

$$\sum_{j \neq i} \frac{\partial \mathcal{H}_i}{\partial q_j} \frac{\partial q_j^{CL}}{\partial p} = -(N-1)\lambda_i s \tag{35}$$

is the additional term in the co-state equation, characterising the strategic interaction among firms, which is not considered by definition in the open-loop solution.

Equation (33) may be rewritten as $\dot{\lambda}_i = (Ns+\rho)\lambda_i - q_i$. This expression, together with $s\lambda(t) = p(t) - c - q_i(t)$ and $\dot{p} = s\left\{A - q_i(t) - \sum_{j \neq i} q_j(t) - p(t)\right\}$, can be substituted into (20) to obtain:

$$\dot{q}_i = [q_i(t) + c](\rho + Ns) - p(t)[\rho + s(N+1)] - s\sum_{j \neq i} q_j(t) \tag{36}$$

with $\dot{q}_i = 0$ in:

$$q_i^*(t) = \frac{p(t)[\rho + s(N+1)] - c(\rho + Ns) + s\sum_{j \neq i} q_j(t)}{\rho + Ns}. \tag{37}$$

Now, imposing the stationarity on price, $\dot{p} = 0$, and substituting into (37), we can write:

$$q_i^* = \frac{(\rho + Ns)\left(A - c - \sum_{j \neq i} q_j\right)}{2\rho + s(2N+1)} \tag{38}$$

that represents the best reply of firm i under the closed-loop solution. On the basis of the above expression, we have:

$$\left.\frac{\partial q_i^*(t)}{\partial q_j(t)}\right|_{CL} = -\frac{\rho + Ns}{2\rho + s(2N+1)}\,; \quad \left|\frac{\partial\left[\partial q_i^*(t)/\partial q_j(t)\right]}{\partial s}\right| = -\frac{\rho}{\left[2\rho + s(2N+1)\right]^2} < 0\,. \tag{39}$$

This entails that the closed-loop best reply has the same qualitative properties of the open-loop one, while the limit behaviour differs, since:

$$\begin{aligned} \lim_{s \to 0} \left.\frac{\partial q_i^*(t)}{\partial q_j(t)}\right|_{CL} &= \lim_{\rho \to \infty} \left.\frac{\partial q_i^*(t)}{\partial q_j(t)}\right|_{CL} = -\frac{1}{2}\,; \\ \lim_{s \to \infty} \left.\frac{\partial q_i^*(t)}{\partial q_j(t)}\right|_{CL} &= \lim_{\rho \to 0} \left.\frac{\partial q_i^*(t)}{\partial q_j(t)}\right|_{CL} = -\frac{N}{2N+1}\,. \end{aligned} \tag{40}$$

Note that $N/(2N+1) > 1/3$ for all $N > 1$, which entails that

Lemma 3 *If price adjustment is instantaneous, the associated best reply at the closed-loop memoryless equilibrium is steeper than the static Cournot best reply for all* $N > 1$.

The above Lemma entails that the limit of the optimal output and price levels at the closed-loop equilibrium cannot coincide with the equilibrium output and price generated by the static game. To ascertain this property, we may invoke symmetry, and rewrite (36) as follows:

$$\dot{q} = \rho\,(c - p + q) + s\,[A - p + q - N\,(p - c)] \tag{41}$$

As in the open-loop case, $\dot{q} = 0$ is a linear relationship between p and q. This, together with $\dot{p} = 0$, which is also a linear function, yields

$$\begin{aligned} p^{CL} &= \frac{A\,[\rho + s\,(N+1)] + N\,(\rho + sN)\,c}{(N+1)\,\rho + (N^2+N+1)\,s}\,; \\ q^{CL} &= \frac{(A-c)\,(\rho + sN)}{(N+1)\,\rho + (N^2+N+1)\,s}\,. \end{aligned} \tag{42}$$

as the unique steady state of the system. The dynamic system can be immediately rewritten in matrix form to verify that the pair $\left\{p^{CL}, q^{CL}\right\}$ is stable in the saddle sense. The proof of this is omitted for the sake of brevity.

Using (42), we may compute:

$$\lim_{s\to 0} q^{CL} = \lim_{\rho\to\infty} q^{CL} = \frac{A-c}{N+1}\,;\ \lim_{s\to 0} p^{CL} = \lim_{\rho\to\infty} p^{CL} = \frac{A+Nc}{N+1}\,; \tag{43}$$

$$\lim_{s\to\infty} q^{CL} = \lim_{\rho\to 0} q^{CL} = \frac{N\,(A-c)}{N^2+N+1}\,;\ \lim_{s\to\infty} p^{CL} = \lim_{\rho\to 0} p^{CL} = \frac{A\,(N+1) + cN^2}{N^2+N+1}\,. \tag{44}$$

Clearly, the output and price levels in (44) are, respectively, larger and smaller than the corresponding equilibrium values for the static game.

From expression (30) and (43), we immediately draw the following implication:

Proposition 4 *If the price is infinitely sticky, both the open-loop equilibrium and the closed-loop memoryless one coincide with perfect competition for all* $N \geq 1$.

Note that this result is not true in general for any closed-loop equilibria. For instance, the feedback equilibrium investigated in Fershtman and Kamien (1987) and Cellini and Lambertini (2004) does collapse into perfect competition in the limit.

Our last result concerns the comparative evaluation of the slope of the best reply across settings. Evaluating (26) against (39), we find:

$$\left| \frac{\partial q_i^*(t)}{\partial q_j(t)} \right|_{CL} > \left| \frac{\partial q_i^*(t)}{\partial q_j(t)} \right|_{OL} \tag{45}$$

for all $N > 1$.

Proposition 5 *The best reply associated with the closed-loop equilibrium is steeper than the best reply associated with the open-loop equilibrium for all* $N > 1$.

The reason is that, when taking into account feedback effects (35) for the closed-loop solution, by definition each firm becomes more sensitive to the rival's behaviour, which makes her best reply steeper than in the open-loop game. For a given intercept of the best reply function, this would imply that firms produce more at the open-loop solution than at the closed-loop one. However, comparing equilibrium outputs in the two settings, we have that $q^{CL} > q^{OL}$, as we know from Fershtman and Kamien (1987) and Cellini and Lambertini (2004). This is due to the fact that, indeed, the intercept of the closed-loop best reply is larger than the intercept of the open-loop best reply, to such an extent that the resulting equilibrium outputs are higher in the closed-loop case.

4 Conclusion

We have investigated the issue of intertemporal strategic interaction in differential games. We have considered a Cournot model with sticky prices where first order conditions do not identify best replies at any time during the game. They only emerge in steady state, where one can check that (i) reaction functions are negatively sloped, (ii) the feedback effects accounted for in the closed-loop solution entail that, in such a case, best replies are steeper than under the open-loop solution; (iii) this notwithstanding, steady state outputs at the closed-loop equilibrium are larger than the corresponding output levels at the open-loop equilibrium, due to the fact that closed-loop best replies are steeper but shifted outwards w.r.t. open-loop best replies. Finally, in the case of infinitely sticky prices, then both types of equilibria replicate the perfectly competitive outcome generated by the static model.

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