Perfect Uncontrollable Differential Games

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Abstract

This paper analyses the time consistency of open-loop equilibria, in the cases of Nash and Stackelberg behaviour. We define a class of games where the strong time-consistency of the open-loop Nash equilibrium associates with the time consistency of the open-loop Stackelberg equilibrium. We label these games as ‘perfect uncontrollable’. We provide one example based on a model of oligopolistic competition in advertising efforts. We also present two oligopoly games where one property holds while the other does not, so that either (i) the open-loop Nash equilibrium is subgame perfect while the stackelberg one is time inconsistent, or (ii) the open-loop Nash and Stackelberg equilibria are only weakly time consistent.

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1 Introduction

The existing literature on simultaneous-move differential games devotes a considerable amount of attention to identifying classes of games where either the feedback or the closed-loop equilibria degenerate into open-loop equilibria. The degeneration means that the Nash-equilibrium time paths of the control variables coincide under the different solution concepts, that is, the optimal paths of control variables depend only on time (and not on states) also under the closed-loop or feedback solution concepts. The interest in the coincidence between the equilibrium path under the different solution concepts is motivated by the following reason. Whenever an open-loop equilibrium is a degenerate closed-loop equilibrium, then the former is also strongly time consistent (or Markovian, or subgame perfect). Therefore, one can rely upon the open-loop equilibrium which, in general, is much easier to derive than closed-loop or feedback ones. Classes of games where this coincidence arises are illustrated in Clemhout and Wan (1974); Reinganum (1982); Mehlmann and Willing (1983); Dockner, Feichtinger and Jørgensen (1985); Fershtman (1987); Fershtman, Kamien and Muller (1992).[1]

Unlike open-loop Nash games, which always generate time consistent (although only weakly) equilibria, Stackelberg open-loop games usually generate time inconsistent equilibria. This has been known ever since Simaan and Cruz (1973a,b).[2] By this, it is meant that, at any intermediate date dur-

[1]For an overview, see Mehlmann (1988) and Dockner, Jørgensen, Long and Sorger (2000, ch. 7).

[2]See Başar and Olsder (1982, 1995[2], ch. 7) and Dockner, Jørgensen, Long and Sorger (2000, ch. 5) for exhaustive overviews.

ing the game, the leader finds it optimal to modify the plan chosen at the outset. This has yielded a large body of literature with economic applications, in particular those concerned with the time inconsistency of optimal monetary or fiscal policy.[3] In a recent contribution, Xie (1997) singled out a property ensuring the time consistency of open-loop Stackelberg equilibria, labelling the games satisfying such a property as *uncontrollable*, in the sense that the leader cannot manipulate the equilibrium at will through his/her control variable.

A striking feature of these two lines of economic research - the time consistency of open-loop Nash equilibria and the time consistency of open-loop Stackelberg equilibria - is that, so far, they haven't overlapped at all. That is, researchers have looked either for those classes of games yielding strongly time consistent open-loop solutions, or for those yielding time consistent Stackelberg open-loop solutions, but not for both at the same time. Of course, this may depend upon the fact that the two solutions are conceptually different. Nevertheless, both for technical reasons and for possible applications, it would surely be desirable to identify classes of games enjoying both properties at the same time.

This is the aim of the present paper. In section 2, we set out with a general illustration of the problem at stake. We label as *perfect uncontrollable game* the game producing (i) strongly time consistent open-loop Nash equilibria *and* (ii) time consistent open-loop Stackelberg equilibria. Then in section 3

[3] This literature is too wide to be duly accounted for here. See, e.g., Kydland and Prescott (1977), Calvo (1978), Barro and Gordon (1983a,b,), Lucas and Stokey (1983) and Cohen and Michel (1988). An overall assessment of this debate can be found in Persson and Tabellini (1990).

we illustrate a duopoly game with advertising (based on a model analysed by Leitmann and Schmitendorf, 1978; and Feichtinger, 1983) to provide one such example. We complete the exposition by briefly examining two cases based upon Cellini and Lambertini (1998) and Simaan and Takayama (1978), where the two properties respectively hold. In the first (section 4), firms accumulate capacity *à la* Ramsey and compete in output levels, while in the second (section 5) firms behave *à la* Cournot under a sticky price dynamics. Possible extensions and concluding comments are in section 6.

2 Setup

Consider a generic differential game, played over continuous time, with $t \in [0, \infty)$.[4] The set of players is $\mathbb{P} \equiv \{1, 2\}$. Moreover, let $x_i(t)$ and $u_i(t)$ define, as usual, the state variable and the control variable pertaining to player i. Assume there exists a prescribed set $\mathcal{U}_i$ such that any admissible action $u_i(t) \in \mathcal{U}_i$. The dynamics of player i's state variable is described by the following:

$$\frac{dx_i(t)}{dt} \equiv \dot{x}_i(t) = f_i\left(\mathbf{x}(t), \mathbf{u}(t)\right) \tag{1}$$

where $\mathbf{x}(t) = (x_1(t), x_2(t))$ is the vector of state variables at time t, and $\mathbf{u}(t) = (u_1(t), u_2(t))$ is the vector of players' actions at the same date, i.e., it is the vector of control variables at time t. That is, in the most general case, the dynamics of the state variable associated with player i depends

[4] One could also consider a finite terminal time T. The specific choice of the time horizon is immaterial to the ensuing analysis, provided that terminal conditions are appropriately defined.

on all state and control variables associated with all players involved in the game. The value of the state variables at $t = 0$ is assumed to be known: $\mathbf{x}(0) = (x_1(0), x_2(0))$.

Each player has an objective function, defined as the discounted value of the flow of payoffs over time. The instantaneous payoff depends upon the choices made by player i as well as its rivals, that is:

$$\pi_i \equiv \pi_i(\mathbf{x}(t), \mathbf{u}(t)). \tag{2}$$

Player i's objective is then, given $u_j(t)$, $j \neq i$:

$$\max_{u_i(\cdot)} J_i \equiv \int_0^\infty \pi_i(\mathbf{x}(t), \mathbf{u}(t)) e^{-\rho t} dt \tag{3}$$

subject to the dynamic constraint represented by the behaviour of the state variables, (1), $u_i(t) \in \mathcal{U}_i$ and initial conditions $\mathbf{x}(0) = (x_1(0), x_2(0))$.

In the literature on differential games, one usually refers to the concepts of *weak* and *strong* time consistency. The difference between these two properties can be outlined as follows:

Definition 1: weak time consistency Consider a game played over $t = [0, \infty)$ and examine the trajectories of the state variables, denoted by $\mathbf{x}(t)$. The equilibrium is weakly time consistent if its truncated part in the time interval $t = [T, \infty)$, with $T \in (0, \infty)$, represents an equilibrium also for any subgame starting from $t = T$, *and* from the vector of initial conditions $\mathbf{x}_T = \mathbf{x}(T)$.

Definition 2: strong time consistency Consider a game played over $t = [0, \infty)$. The equilibrium is strongly time consistent, if its truncated part is an equilibrium for the subgame, independently of the conditions regarding state variables at time T, $\mathbf{x}(T)$.

Strong time consistency requires the ability on the part of each player to account for the rival's behaviour at any point in time, i.e., it is, in general, an attribute of closed-loop equilibria, and corresponds to subgame perfectness. Weak time consistency is a milder requirement and does not ensure, in general, that the resulting Nash equilibrium be subgame perfect.[5]

Now consider the Stackelberg differential game, and assume player i is the follower (we shall subsequently address the leader's problem). The Hamiltonian of player i is:[6]

$$\begin{aligned}\mathcal{H}_i(\mathbf{x}(t),\mathbf{u}(t)) \equiv\ & e^{-\rho t}\left[\pi_i(\mathbf{x}(t),\mathbf{u}(t)) + \lambda_{ii}(t)\cdot f_i(\mathbf{x}(t),\mathbf{u}(t)) + \right. \\ & \left. + \lambda_{ij}(t)\cdot f_j(\mathbf{x}(t),\mathbf{u}(t))\right], \qquad (4)\end{aligned}$$

where $\lambda_{ij}(t) = \mu_{ij}(t)e^{\rho t}$ is the co-state variable (evaluated at time t) that firm i associates with the state variable $x_j(t)$.

In the remainder of the paper, we shall focus on first order conditions alone, under the assumption that sufficiency (second order) conditions are met. This will be apparent in the examples illustrated in the next sections. Moreover, we shall adopt the following conventional notation: steady state values of controls and states are identified by superscript s; optimal controls or states satisfying the necessary conditions are starred.

[5]For a more detailed analysis of these issues, see Dockner *et al.* (2000, Section 4.3, pp. 98-107); see also Başar and Olsder (1982, 1995^2, ch. 6).

[6]By the definition, the follower's Hamiltonian function is the same as in the Nash game (see, e.g., Dockner *et al.*, 2000).

The first order condition (FOC) on the control variable $u_i(t)$ is:[7]

$$\frac{\partial \mathcal{H}_i\left(\mathbf{x}^*(t), \mathbf{u}^*(t)\right)}{\partial u_i} = 0,\ i,\ j = 1, 2; \tag{5}$$

$$\Leftrightarrow \frac{\partial \pi_i\left(\mathbf{x}^*(t), \mathbf{u}^*(t)\right)}{\partial u_i} + \lambda_{ii}(t)\frac{\partial f_i\left(\mathbf{x}^*(t), \mathbf{u}^*(t)\right)}{\partial u_i} + \lambda_{ij}(t)\frac{\partial f_j\left(\mathbf{x}^*(t), \mathbf{u}^*(t)\right)}{\partial u_i} = 0$$

and the adjoint equations concerning the dynamics of state and co-state variables are as follows:

$$-\frac{\partial \mathcal{H}_i\left(\mathbf{x}^*(t), \mathbf{u}^*(t)\right)}{\partial x_j} = \frac{\partial \lambda_{ij}(t)}{\partial t} - \rho \lambda_{ij}\left(t\right)\ ,\ i,\ j = 1, 2; \tag{6}$$

$$\Leftrightarrow \quad \frac{\partial \lambda_{ij}(t)}{\partial t} = \rho \lambda_{ij}\left(t\right) - \frac{\partial \pi_i\left(\mathbf{x}^*(t), \mathbf{u}^*(t)\right)}{\partial x_j} + \\ -\lambda_{ii}(t)\frac{\partial f_i\left(\mathbf{x}^*(t), \mathbf{u}^*(t)\right)}{\partial x_j} - \lambda_{ij}(t)\frac{\partial f_j\left(\mathbf{x}^*(t), \mathbf{u}^*(t)\right)}{\partial x_j}$$

They have to be considered alongside with the initial conditions $\mathbf{x}(0) = \left(x_1\left(0\right), x_2\left(0\right)\right)$ and the transversality conditions, which set the final value (at $t = \infty$) of the state and/or co-state variables. In problems defined over an infinite time horizon, one usually sets:[8]

$$\lim_{t \to \infty} e^{-\rho t} \lambda_{ij}(t) \cdot x_j(t) = 0,\ i, j = 1, 2. \tag{7}$$

For simplicity, we consider the case where only one state and one control are associated with every single player. If the evolution of the state variable $x_i(t)$ depends only upon $\{x_i(t), u_i(t)\}$, i.e., it is independent of $u_j(t)$ and $x_j(t)$ and (1) reduces to $\dot{x}_i(t) = f_i\left(x_i(t), u_i(t)\right)$, then the game exhibits separate

[7] The indication of exponential discounting is omitted for brevity.

[8] Otherwise, if $t \in [0, T]$, a different transversality condition applies. For instance, if the value of $x_j\left(T\right)$ differs from zero, then one may set $\lambda_{ij}\left(T\right) = 0$. See section 3.

dynamics, and one can set $\lambda_{ij}(t) = 0$ for all $j \neq i$, which entails that the Hamiltonian of player i can be written by taking into account the dynamics of i's state variable only.

Before proceeding to the analysis of the leader's problem, we would like to clarify that there exist two possible approaches to the description of strategic interaction between players in a Stackelberg game. The first consists in asking the leader (player j) to choose a proper reaction function $u_j^*(u_i^*(t))$ which fully specifies the leader's best reply to the follower's optimal behaviour at any instant t during the game. Alternatively, following Dockner *et al.* (2000, chapter 5.3), one may suppose that the leader can announce to the follower the policy $u_i^L(\mathbf{x}(t))$ that she (the leader) will use throughout the game, defined in terms of the states only. The follower, taking the rule $u_i^L(\mathbf{x}(t))$ as given, determines the reaction function $u_i^F\left(u_i^L(\mathbf{x}(t))\right)$ to maximise his payoff. The problem of the leader is then to choose, among all the admissible rules $u_i(\mathbf{x}(t))$, that particular $u_i^L(\mathbf{x}(t))$ that maximises her payoff, given the follower's bets reply and all the additional relevant constraints. Dockner *et al.* (2000, chapter 5.3) provide the solution to this problem by confining their attention to games where only one state variable appears and therefore one can write $u_i^L = w + zx$, where $w, z \in \mathbb{R}$. The leader's optimization problem amounts to choosing w and z once and for all. A major objection to this approach is that, indeed, this is not a game where both players have Markovian state information. Moreover, the fact that the leader chooses w and z at the outset and keep them constant throughout the game is responsible for the time consistency characterising the resulting Stackelberg equilibrium (for a more extensive discussion of these issues, see Dockner *et al.*, 2000, p. 135).

In the present paper, we will take the first route. From Simaan and Cruz (1973a,b), we know that, in general, open-loop Stackelberg games yield time inconsistent equilibria. However, if the game structure meets a specific requirement, the Stackelberg open-loop equilibrium turns out to be time consistent. To illustrate this requirement, one has to proceed as follows. From (5) one obtains the instantaneous best reply of player i, which can be differentiated with respect to time to yield the dynamic equation of the control variable $u_i(t)$. Moreover, given (1), the first order condition (5) will contain the co-state variable $\lambda_{ii}(t)$ associated with the kinematic equation of the state variable $x_i(t)$. Therefore, (5) can be solved w.r.t. $\lambda_{ii}(t)$ so as to yield:

$$\lambda_{ii}(t) = -\left[\frac{\partial \pi_i\left(\mathbf{x}^*(t), \mathbf{u}^*(t)\right)}{\partial u_i} + \lambda_{ij}(t)\frac{\partial f_j\left(\mathbf{x}^*(t), \mathbf{u}^*(t)\right)}{\partial u_i}\right] / \frac{\partial f_i\left(\mathbf{x}^*(t), \mathbf{u}^*(t)\right)}{\partial u_i} \tag{8}$$

If the expression on the r.h.s. in (8) contains the leader's control variable $u_j(t)$, then the open-loop Stackelberg strategies are bound to be time inconsistent, in that the leader can control the follower's state dynamics by manoeuvring $u_j(t)$. In such a case, the game is *controllable* by the leader, who cannot resist the temptation to renege any initial plans later on during the game. If, instead, $\lambda_{ii}(t)$ does not depend upon $u_j(t)$, then the game is *uncontrollable* by the leader, and the resulting open-loop Stackelberg equilibrium strategies are time consistent (Xie, 1997; see also Dockner *et al.*, 2000, ch. 5).

That is, uncontrollability relies upon the following property:

$$\frac{\partial \lambda_{ii}(t)}{\partial u_j(t)} = 0,\ i,\ j = 1, 2;\ j \neq i \tag{9}$$

which must hold for (8) (as well as $\partial\lambda_{ij}(t)/\partial u_j(t) = 0$, if $\lambda_{ij}(t) \neq 0$). This amounts to saying that $\lambda_{ii}(t)$ does not depend on $u_j(t)$, i.e., the leader cannot affect the co-state variable of the follower, and the resulting Stackelberg solution is time consistent. Since $\lambda_{ii}(t)$ in (8) comes from the solution of (5), we can say that (9) is equivalent to:

$$\frac{\partial^2 \mathcal{H}_i\left(\mathbf{x}^*(t), \mathbf{u}^*(t)\right)}{\partial u_i \partial u_j} = 0,\ i,\ j = 1, 2;\ j \neq i. \tag{10}$$

For completeness, we may briefly summarise the issue of strong time consistency of open-loop Nash equilibria. If

$$\frac{\partial^2 \mathcal{H}_i\left(\mathbf{x}^*(t), \mathbf{u}^*(t)\right)}{\partial u_i \partial x_j} = 0,\ i,\ j = 1, 2;\ j \neq i \tag{11}$$

either immediately from the Hamiltonian function, or by appropriate substitutions from the co-state equations into the first order conditions, then the optimal controls are independent of states and the open-loop equilibrium is subgame (or Markov) perfect since it is strongly time consistent. The property whereby the FOCs on controls are independent of states and initial conditions after replacing the optimal values of the co-state variables is known as *state-redundancy*, and the game itself as *state-redundant* or *perfect* (Fershtman, 1987; Mehlmann, 1988, ch. 4). Of course condition (10) in general does not coincide with condition (11). Whenever (10) and (11) are simultaneously met within the same game, then the game itself is a *perfect uncontrollable game*.

Accordingly, we may formulate the following:

Proposition 1 (State-Redundancy) *If a differential game is perfect, then its open-loop Nash equilibrium is strongly time consistent (or subgame perfect).*

Proposition 2 (Uncontrollability) *If a differential game is uncontrollable by all of the players involved, then all of its open-loop Stackelberg equilibria are time consistent.*

Proposition 1 says that state-redundancy is necessary and sufficient to generate Markov-perfect open-loop Nash equilibria. However, note that the uncontrollability is necessary but not sufficient to generate subgame (or Markov) perfect equilibria in the open-loop Stackelberg game. The reason is that uncontrollability is unrelated to feedback effects, i.e., the presence of feedback effects throughout the game may prevent an uncontrollable game to yield strongly time consistent Stackelberg equilibria under the open-loop information structure. Put together, Propositions 1-2 imply the following relevant Corollary:

Corollary 3 *Consider an open-loop Stackelberg game. If it is both uncontrollable and perfect, then its Stackelberg equilibria are strongly time consistent (or subgame perfect). Otherwise, if it is perfect but controllable by at least one of the players, then the open-loop Stackelberg equilibrium with that player leading is bound to be time inconsistent.*

The above Corollary says that uncontrollability and state-redundancy must hold together in order for the open-loop Stackelberg behaviour to generate Markov-perfect solutions. More explicitly, in the Stackelberg case, state-redundancy is necessary but not sufficient to yield Markov-perfectness.

Observing (10) and (11), a further Remark emerges:

Remark 4 *The additive separability of each player's Hamiltonian function w.r.t. state and control variables is sufficient to ensure that the game is per-*

fect and uncontrollable and all of its open-loop Nash and Stackelberg equilibria are strongly time consistent (or subgame perfect).

The proof is trivial, in that additive separability implies that the first order condition of each player is independent of both the rival's control and state variables. However, it is worth stressing that additive separability is sufficient but by no means necessary to make the game perfect and uncontrollable, as it will become clear by examining the duopoly model illustrated in the next section.[9]

3 A perfect uncontrollable game

As in Leitmann and Schmitendorf (1978; LS henceforth) and Feichtinger (1983), we have a non cooperative differential game over $t \in [0, T]$ between two firms, 1 and 2, choosing their respective advertising efforts $u_i(t)$ to maximise their individual discounted payoff:

$$J_i = \int_0^T [p_i x_i(t) - u_i(t)] e^{-\rho t} dt, \; i = 1, 2 \tag{12}$$

where $x_i(t)$, firm i's market share, is a state variable evolving according to:

$$\dot{x}_i = -\beta_i x_i(t) + u_i(t) - \frac{1}{2} c_i u_i^2(t) - k_i x_i(t) u_j(t), \; i, j = 1, 2; \; j \neq i. \tag{13}$$

p_i, β_i, c_i and k_i are positive parameters. In particular, $\beta_i \in [0, 1]$ is the decay rate of firm i's market share,[10] while k_i measures the spillover from

[9] The analysis of dynamic oligopoly interaction has generated a relatively wide literature that cannot be exhaustively accounted for here. To the best of our knowledge, the earliest differential duopoly game can be found in Clemhout, Leitmann and Wan (1971).

[10] To ensure that $x_i(t) + x_j(t) \leq 1$, one has to impose $\beta_i \geq 1/(2c_i) + 1/(2c_j)$ (Lemma 1 in LS, p. 646).

the rival's investment $u_j(t)$ in proportion to firm i's current demand $x_i(t)$. That is, $k_i x_i(t)$ is the correspondent of what is usually called 'absorptive capacity' in the literature on research and development[11] (except that, in the present case, the effect is negative because it entails 'business stealing'). The factor $e^{-\rho t}$ discounts future gains, and the discount rate ρ is assumed to be constant and common to all players.[12] In addition to (13), we also adopt the further constraint whereby $u_i(t) \in [0, 1/c_i]$, which amounts to saying that there is an upper bound to the advertising investment of firm i, $i = 1, 2$. This restriction remains to be checked ex post, once we are in a position to determine the features of the steady state equilibrium.

The Hamiltonian of firm i is:

$$\mathcal{H}_i(\mathbf{x}(t), \mathbf{u}(t)) = e^{-\rho t}\left\{[p_i x_i(t) - u_i(t)] + \lambda_{ii}(t)\dot{x}_i(t) + \lambda_{ij}(t)\dot{x}_j(t)\right\}, \quad (14)$$

with i, $j = 1, 2$; $j \neq i$, where the scrap value at the terminal date T is set equal to zero for the sake of simplicity, and without further loss of generality. We know from LS that the Nash open-loop solution is a degenerate closed-loop one, i.e., there exists a state-independent feedback control for each player. Using

$$\begin{aligned} \dot{\lambda}_{ij}(t) = \lambda_{ij}(t)\left[k_j u_i^*(t) + \beta_j + k_j x_j^*(t)\frac{\partial u_i^*(t)}{\partial x_j} + \rho\right] + \\ -\lambda_{ii}(t)[1 - c_i u_i^*(t)]\frac{\partial u_i^*(t)}{\partial x_j} = 0 \end{aligned} \quad (15)$$

[11]See, e.g., Kamien and Zang (2000).

[12]In the original version of the LS model, the discount rate is nil. As we show in the remainder, introducing positive discounting, while making the analysis a bit more realistic at least in economic terms, does not modify significantly the conclusions.

and

$$\frac{\partial u_i^*(t)}{\partial x_j(t)} = -\frac{\lambda_{ij}(t)\,k_j}{\lambda_{ii}(t)\,c_i}\,,\; i,\, j = 1, 2;\; j \neq i. \tag{16}$$

one finds that the problem admits the solution $\lambda_{ij}(t) = 0$ for $j \neq i$, at all t. This, in turn, entails that the first order condition:[13]

$$\frac{\partial \mathcal{H}_i\left(\mathbf{x}^*(t), \mathbf{u}^*(t)\right)}{\partial u_i} = 0 \Leftrightarrow u_i^*(t) = \frac{\lambda_{ii}(t) - 1 - \lambda_{ij}k_j x_j^*(t)}{c_i\lambda_{ii}(t)} \tag{17}$$

can be rewritten as follows:

$$\begin{aligned} u_i^*(t) &= \frac{\lambda_{ii}(t) - 1}{c_i\lambda_{ii}(t)} \geq 0 \text{ iff } \lambda_{ii}(t) \geq 1; \\ u_i^*(t) &= 0 \text{ otherwise; } i = 1, 2 \end{aligned} \tag{18}$$

At this point, from (18) there emerges the property $\partial u_i^*(t)/\partial x_j(t) = 0$, so that the open-loop solution is a degenerate closed-loop one, and yields a Markov equilibrium. The aforementioned property is equivalent to (11). We can summarise the above discussion in the following:

Lemma 5 (Leitmann and Schmitendorf, 1978) *The open-loop Nash solution of the LS game is strongly time consistent. Therefore, the LS game is perfect.*

It is worth stressing that the above method is that employed by Mehlmann (1988) in revisiting the LS model. The alternative method adopted by Leitmann and Schmitendorf consists in showing that the open-loop controls depend only upon time but not states and initial conditions. In both cases,

[13]Second order conditions, which are omitted for brevity, can be shown to hold by using, e.g., Arrow's sufficiency conditions (Arrow, 1968). For global sufficiency conditions, which apply also in the case of feedback solutions, see Stalford and Leitmann (1973).

one ascertains that the feedback effects at any instant t during the game are endogenously nil, and therefore the open-loop Nash equilibrium is indeed Markovian, i.e., it is the collapse of a closed-loop equilibrium.

To complete the characterisation of the open-loop Markovian equilibrium, from (18) we may obtain (the indication of time is omitted henceforth):

$$\dot{u}_i^* = \frac{\dot{\lambda}_{ii}}{c_i \lambda_{ii}^2} \tag{19}$$

provided $\lambda_{ii} \geq 1$. The dynamics of λ_{ii} is given by the following co-state equation, derived under the open-loop information structure:

$$-\frac{\partial \mathcal{H}_i (\mathbf{x}^*, \mathbf{u}^*)}{\partial x_i} = \dot{\lambda}_{ii} - \rho \lambda_{ii} \Leftrightarrow \dot{\lambda}_{ii} = \lambda_{ii} \left(\rho + \beta_i + k_i u_j^* \right) - p_i \, . \tag{20}$$

The transversality condition is $\lambda_{ij}(T) = 0$, $i, \, j = 1, 2$.

Expression (18) can be rearranged to obtain:

$$\lambda_{ii} = \frac{1}{1 - c_i u_i^*} \, . \tag{21}$$

Now (20-21) can be substituted into (19) to obtain:

$$\dot{u}_i^* = \frac{(1 - c_i u_i^*) \left[\rho + \beta_i - p_i (1 - c_i u_i^*) + k_j u_i^* \right]}{c_i}, \; i, \, j = 1, 2; \, j \neq i. \tag{22}$$

The steady state values of optimal controls and states can be found by solving the system $\left(\dot{x}_i = 0; \dot{u}_i^* = 0 \right)$ w.r.t. $\mathbf{x}$ and $\mathbf{u}$. To simplify the exposition and focus the attention on the fundamental properties of our analysis, at this point we may introduce some symmetry conditions, whereby $c_i = c_j = c$, $k_i = k_j = k$, $p_i = p_j = p$, $\beta_i = \beta_j = \beta$, $u_i = u_j = u$ and $x_i = x_j = x$. In this way, we confine to symmetric steady state equilibria.[14] The stationarity

[14]It can also be shown that indeed a unique steady state equilibrium exists, and it is symmetric, even without imposing the above symmetry conditions.

condition $\dot{x} = 0$ yields:

$$x^s = \frac{u^* (2 - cu^*)}{2 (\beta + ku^*)} \tag{23}$$

which can be plugged into (22) so that $\dot{u}^* = 0$ in:

$$u_a \equiv \frac{1}{c}\,;\; u_b \equiv \frac{p - \beta - \rho}{cp + k}\,. \tag{24}$$

Note that $u_b < u_a$ always, and $u_b > 0$ iff $p > \beta + \rho$. The latter condition means that, in order for the steady state advertising effort to be positive, the marginal value that the firm attaches to any additional customer (p) must be higher than the sum of depreciation and discount rates.

As a last step, we evaluate the sign of the determinant of the Jacobian matrix of the dynamic system $\left(\dot{x}; \dot{u}^*\right)$ to show that indeed u_b is the optimal solution. The determinant is:

$$\Delta \equiv \frac{\partial \dot{x}}{\partial x} \cdot \frac{\partial \dot{u}}{\partial u} - \frac{\partial \dot{x}}{\partial u} \cdot \frac{\partial \dot{u}}{\partial x} \tag{25}$$

which in (x^s, u_b) corresponds to:

$$\Delta|_{x^s, u_b} = -\frac{[cp\beta + k (p - \rho)]\,[k + c (\beta + \rho)]}{c (cp + k)} < 0 \tag{26}$$

given $p > \beta + \rho$. That is, the positivity of the advertising investment is sufficient to ensure that $(x^s, u_b)$is a saddle point. The expression of the co-state variable (21) simplifies as follows:

$$\lambda_{ii} = \frac{cp + k}{k + c (\beta + \rho)} > 1 \text{ for all } p > \beta + \rho. \tag{27}$$

Conversely, the solution (x^s, u_a) is unstable, since

$$\Delta|_{x^s, u_a} = \frac{(c\beta + k)\,[k + c (\beta + \rho)]}{c^2} > 0. \tag{28}$$

We may summarise the foregoing analysis in:

Proposition 6 *The open-loop Nash game yields a unique saddle point where the symmetric steady state values of the Markovian controls are:*

$$u^s = \frac{p - \beta - \rho}{cp + k}.$$

The corresponding steady state level of the state variable is:

$$x^s = \frac{(p - \beta - \rho)\left[2k + c\left(p + \beta + \rho\right)\right]}{2\left(c\beta + k\right)\left[cp\beta + k\left(p - \rho\right)\right]}.$$

Now we move on to the Stackelberg open-loop solution of the same game, which can be shown to be subgame (Markov) perfect as well. Before doing that, a preliminary step can be taken here to show the following property:

Lemma 7 *At any instant during the game,* $\partial u_i^* / \partial u_j < 0$*, i.e., optimal controls are strategic substitutes.*

Proof. Abandon the symmetry assumptions we have just considered in order to quickly characterise the steady state of the system, and examine again (22). This dynamic equation entails that $\dot{u}_i^* = 0$ in $u_i^* = 1/c_i$, which, as we know, can be disregarded, and

$$u_i^* = \frac{p_i - \beta_i - \rho - k_i u_j}{c_i p_i} \tag{29}$$

defining the instantaneous best reply (or reaction) function of firm i to the strategy of firm j. Now observe that

$$\frac{\partial u_i^*}{\partial u_j} = -\frac{k_i}{c_i p_i} < 0 \tag{30}$$

which entails that reaction functions are negatively sloped. That is, the game takes place in strategic substitutes, in the sense that any increase in firm j's control triggers a decrease in firm i's control, and conversely. The stage game at any t is therefore submodular.[15] ■

Lemma 7 will help us characterising the qualitative properties of the open-loop Stackelberg equilibrium.

3.1 The Stackelberg solution of the LS game

To simplify exposition, we stipulate that firm 1 is the leader and firm 2 is the follower. The follower maximizes

$$\mathcal{H}_2\left(\mathbf{x}(t), \mathbf{u}(t)\right) = e^{-\rho t}\left\{\left[p_2 x_2\left(t\right) - u_2\left(t\right)\right] + \lambda_{22}\left(t\right) \dot{x}_2\left(t\right) + \lambda_{21}\left(t\right) \dot{x}_1\left(t\right)\right\}, \tag{31}$$

where

$$\begin{aligned} \dot{x}_1\left(t\right) &= -\beta_1 x_1\left(t\right) + u_1\left(t\right) - \tfrac{1}{2} c_1 u_1^2\left(t\right) - k_1 x_1\left(t\right) u_2\left(t\right); \\ \dot{x}_2\left(t\right) &= -\beta_2 x_2\left(t\right) + u_2\left(t\right) - \tfrac{1}{2} c_2 u_2^2\left(t\right) - k_2 x_2\left(t\right) u_1\left(t\right). \end{aligned} \tag{32}$$

The FOCs for the open-loop solution are:[16]

$$\frac{\partial \mathcal{H}_2\left(\mathbf{x}^*, \mathbf{u}^*\right)}{\partial u_2} = \lambda_{22}\left[1 - c_2 u_2^*\right] - 1 - \lambda_{21} k_1 x_1^* = 0 \Leftrightarrow \tag{33}$$

$$u_2^* = \frac{\lambda_{22} - 1 - \lambda_{21} k_1 x_1^*}{c_2 \lambda_{22}} \tag{34}$$

$$\frac{\partial \mathcal{H}_2\left(\mathbf{x}^*, \mathbf{u}^*\right)}{\partial x_2} = \dot{\lambda}_{22} - \rho \lambda_{22} \Leftrightarrow \dot{\lambda}_{22} = \lambda_{22}\left(k_2 u_1^* + \beta_2 + \rho\right) - p_2 \tag{35}$$

$$\frac{\partial \mathcal{H}_2\left(\mathbf{x}^*, \mathbf{u}^*\right)}{\partial x_1} = \dot{\lambda}_{21} - \rho \lambda_{21} \Leftrightarrow \dot{\lambda}_{21} = \lambda_{21}\left(k_1 u_2^* + \beta_1 + \rho\right) \tag{36}$$

[15] The label of strategic substitutability/complementarity dates back to Bulow, Geanakoplos and Klemperer (1985). On supermodular/submodular games, see Amir (1996) and Vives (1999), *inter alia*.

[16] The indication of time and exponential discounting is omitted for brevity.

From (33), observe that

$$\frac{\partial \lambda_{22}}{\partial u_1} = \frac{\partial^2 \mathcal{H}_2\left(\mathbf{x}^*, \mathbf{u}^*\right)}{\partial u_2 \partial u_1} = 0 \tag{37}$$

which immediately entails the following result:

Lemma 8 *The Stackelberg LS game is uncontrollable by the leader. Therefore, the open-loop Stackelberg solution is time consistent.*

Lemmas 5 and 8 imply:

Proposition 9 *The LS model is a perfect uncontrollable game. Therefore, all of its Nash and Stackelberg open-loop equilibria are subgame perfect.*

Just to complete the solution of the model, we proceed to examine the leader's problem, where (34-35-36) appear as additional constraint that the leader must account for in choosing his own optimal advertising plan. Accordingly, firm 1's Hamiltonian is:

$$\mathcal{H}_1\left(\mathbf{x}(t), \mathbf{u}(t)\right) = e^{-\rho t}\Big\{\left[p_1 x_1(t) - u_1(t)\right] + \lambda_{11}(t)\,\dot{x}_1(t) + \tag{38}$$

$$+\lambda_{12}(t)\,\dot{x}_2(t) + \varpi_1(t)\,\dot{\lambda}_{22}(t) + \varpi_2(t)\,\dot{\lambda}_{21}(t)\Big\} \tag{39}$$

where $\varpi_1(t)$ and $\varpi_2(t)$ are the adjoint variables (in current value) attached to the follower's co-state dynamics (35-36), with $\lambda_{22}(t)$ and $\lambda_{21}(t)$ acting as additional state variables in the leader's maximization problem. Using also (34), the Hamiltonian of the leader is as follows:[17]

$$\mathcal{H}_1\left(\mathbf{x}, \mathbf{u}\right) = e^{-\rho t}\left\{\left[p_1 x_1 - u_1\right] + \lambda_{11}\left[-\beta_1 x_1 + u_1 - \frac{1}{2}c_1 u_1^2 - k_1 x_1 \frac{\lambda_{22} - 1 - \lambda_{21} k_1 x_1}{c_2 \lambda_{22}}\right] + \right.$$

[17]Again, the indication of time end exponential discounting is omitted henceforth.

$$+\lambda_{12}\left[-\beta_2 x_2+\frac{\lambda_{22}-1-\lambda_{21}k_1x_1}{c_2\lambda_{22}}-\frac{1}{2}c_2\left(\frac{\lambda_{22}-1-\lambda_{21}k_1x_1}{c_2\lambda_{22}}\right)^2-k_2x_2u_1\right]+$$

$$+\varpi_1\left[\lambda_{22}\left(k_2u_1+\beta_2+\rho\right)-p_2\right]+\varpi_2\lambda_{21}\left(k_1\frac{\lambda_{22}-1-\lambda_{21}k_1x_1}{c_2\lambda_{22}}+\beta_1+\rho\right)\Bigg\} \tag{40}$$

Now we can take the FOC w.r.t. u_1 :

$$\frac{\partial\mathcal{H}_1\left(\mathbf{x}^*,\mathbf{u}^*\right)}{\partial u_1} = -1+\lambda_{11}-\lambda_{11}c_1u_1^*-k_2\left(\lambda_{12}x_2^*-\varpi_1\lambda_{22}\right)=0\Leftrightarrow \tag{41}$$

$$\lambda_{11} = -\frac{1+k_2\left(\lambda_{12}x_2^*-\varpi_1\lambda_{22}\right)}{c_1u_1^*-1} \tag{42}$$

Moreover, from (41) we also obtain:

$$u_1^*=\frac{\lambda_{11}-1-k_2\left(\lambda_{12}x_2^*-\varpi_1\lambda_{22}\right)}{\lambda_{11}c_1} \tag{43}$$

from which we obtain the kinematic equation of the leader's optimal investment:

$$\begin{aligned}\dot{u}_1 &= \frac{1}{\lambda_{11}^2c_1}\left[\dot{\lambda}_{11}\left(1+k_2\left(\lambda_{12}x_2^*-\varpi_1\lambda_{22}\right)\right)+\right. \\ &\left.+\lambda_{11}k_2\left(\varpi_1(t)\,\dot{\lambda}_{22}+\dot{\varpi}_1(t)\,\lambda_{22}-x_2^*\dot{\lambda}_{12}-\dot{x}_2\lambda_{12}\right)\right.\end{aligned} \tag{44}$$

The co-state equations are:

$$-\frac{\partial\mathcal{H}_1\left(\mathbf{x}^*,\mathbf{u}^*\right)}{\partial x_1}=\dot{\lambda}_{11}-\rho\lambda_{11}\Leftrightarrow$$

$$\begin{aligned}\dot{\lambda}_{11} &= \frac{1}{\lambda_{22}^2c_2}\left[k_1\left(\lambda_{12}\lambda_{21}+\lambda_{11}\lambda_{22}\left(\lambda_{22}-1\right)\right)+\lambda_{22}^2c_2\left(\lambda_{11}\left(\beta_1+\rho\right)-p_1\right)+\right. \\ &\left.+\lambda_{21}k_1^2\left(x_1^*\left(\lambda_{12}\lambda_{21}-2\lambda_{11}\lambda_{22}\right)+\lambda_{21}\lambda_{22}\varpi_2\right)\right]\end{aligned} \tag{45}$$

$$-\frac{\partial\mathcal{H}_1\left(\mathbf{x}^*,\mathbf{u}^*\right)}{\partial x_2}=\dot{\lambda}_{12}-\rho\lambda_{12}\Leftrightarrow\dot{\lambda}_{12}=\lambda_{12}\left(k_2u_1^*+\beta_2+\rho\right) \tag{46}$$

$$-\frac{\partial \mathcal{H}_1(\mathbf{x}^*, \mathbf{u}^*)}{\partial \lambda_{22}} = \dot{\varpi}_1 - \rho\varpi_1 \Leftrightarrow$$

$$\begin{aligned}\dot{\varpi}_1 &= -\frac{1}{\lambda_{22}^3 c_2}\left[\lambda_{12}(1+\lambda_{21}k_1x_1)^2 + \lambda_{22}\left(\varpi_1\lambda_{22}^2 c_2(k_2u_1^* + \beta_2) + \right.\right.\\ &\left.\left. +k_1(\varpi_2\lambda_{21} - \lambda_{11}x_1^*)(1+\lambda_{21}k_1x_1^*)\right)\right] \end{aligned} \tag{47}$$

$$-\frac{\partial \mathcal{H}_1(\mathbf{x}^*, \mathbf{u}^*)}{\partial \lambda_{21}} = \dot{\varpi}_2 - \rho\varpi_2 \Leftrightarrow$$

$$\begin{aligned}\dot{\varpi}_2 &= -\frac{1}{\lambda_{22}^2 c_2}\left[k_1(\lambda_{12}x_1^* - \varpi_2\lambda_{22}(\lambda_{22}-1)) - \varpi_2\lambda_{22}^2 c_2\beta_1 + \right.\\ &\left. k_1^2 x_1^*(2\varpi_2\lambda_{21}\lambda_{22} + x_1^*(\lambda_{12}\lambda_{21} - \lambda_{11}\lambda_{22}))\right] \end{aligned} \tag{48}$$

Now (32) and (35), together with (45-47) can be inserted into (44) to rewrite the leader's control dynamics. We look for possible steady state(s), and, imposing the stationarity condition upon the follower's co-state dynamics, we obtain:

$$\begin{aligned}\dot{\lambda}_{21} = 0 \text{ in } \lambda_{21} = 0\\ \dot{\lambda}_{22} = 0 \text{ in } \lambda_{22} = \frac{p_2}{k_2u_1^* + \beta_2 + \rho}\end{aligned} \tag{49}$$

Note that $\lambda_{21} = 0$ as in the Nash game. Then, using (42) and imposing $\dot{\lambda}_{12} = 0$, we obtain $\lambda_{12} = 0$.

In order to further simplify matters, we introduce a useful (albeit arguable) set of symmetry conditions: $c_2 = c_1 = c$, $k_2 = k_1 = k$, $p_2 = p_1 = p$ and $\beta_2 = \beta_1 = \beta$. Now, imposing $\dot{\varpi}_1 = 0$, we have:

$$\varpi_1 = \frac{kx_1^*(ku_1^* + \beta + \rho)}{p\left[cp(1-cu_1^*)(ku_1^* + \beta) + k^2x_1^*(ku_1^* + \beta + \rho)\right]} \tag{50}$$

while the value of ϖ_2 is irrelevant, since we may write:

$$\begin{aligned}\dot{u}_1 &= \frac{1}{c^2p(ku_1^* + \beta)}\left\{(cu_1^* - 1)\left[(ku_1^* + \beta)(k - cp)(ku_1^* + \beta + \rho + \right.\right.\\ &\left.\left. +p(cu_1^* - 1)) + k^2px_1^*(ku_1^* + \beta + \rho)\right]\right\} \end{aligned} \tag{51}$$

where ϖ_2 does not appear. Imposing the stationarity condition $\dot{x}_i = 0$, $i = 1, 2$, upon the system of state equations (32), we obtain the expressions of the optimal states as a function of the leader's optimal control:

$$\begin{aligned} x_1^s &= \frac{cpu_1^* (2 - cu_1^*)}{2 [cp\beta - k^2 u_1^* + k (p - \beta - \rho)]} \\ x_2^s &= \frac{p^2 - (ku_1^* + \beta + \rho)^2}{2cp^2 (ku_1^* + \beta)} \end{aligned} \tag{52}$$

As a last step, one should solve $\dot{u}_1 = 0$ w.r.t. u_1.[18] Without going through numerical calculations, we may rely on Lemma 7 to state what follows:

Proposition 10 *Since the game exhibits decreasing best reply functions at any instant, the Stackelberg solution entails that the leader invests more than the follower.*

This claim has a definite Cournot flavour: with strategic substitutes, hierarchical play involves a first-mover advantage, and a corresponding second-mover disadvantage. In fact, it can be shown by numerical calculations that the resulting market shares are, respectively, larger (for the leader) and smaller (for the follower) than in the steady state of the Nash game.

4 The Ramsey oligopoly game

Here, we illustrate a game which is state-redundant (or perfect), but controllable. Consider a market where two single-product firms, labelled as 1 and

[18] The expression for $\dot{u}_1$ is cubic, with only one solution belonging to the set of real positive numbers. Details are available from the authors upon request.

2, offer homogeneous products over $t \in [0, \infty)$. At any time t, the inverse market demand function is:

$$p(t) = A - B\left[q_1(t) + q_2(t)\right]. \quad (53)$$

Production requires investment in physical capital, which accumulates over time to create capacity. At any t, the output level is $y_i(t) = f(\kappa_i(t))$, with $f' \equiv \partial f(\kappa_i(t))/\partial \kappa_i(t) > 0$ and $f'' \equiv \partial^2 f(\kappa_i(t))/\partial \kappa_i^2(t) < 0; i = 1, 2$.

A reasonable assumption is that $q_i(t) \leq y_i(t)$, that is, the level of sales is at most equal to the quantity produced. Excess output is reintroduced into the production process yielding accumulation of capacity according to the following process (Ramsey, 1928):

$$\frac{d\kappa_i(t)}{dt} = f(\kappa_i(t)) - q_i(t) - \delta\kappa_i(t),\ i = 1, 2, \quad (54)$$

where δ denotes the rate of depreciation of capital. In order to simplify further the analysis, suppose that unit variable cost is constant and equal to zero. The cost of capital is represented by the opportunity cost of intertemporal relocation of unsold output. Firm i's instantaneous profits i are

$$\pi_i\left(\mathbf{q}\left(t\right)\right) \equiv p(t)q_i(t) = \left[A - B\left(q_i(t) + q_j(t)\right)\right] q_i(t),\ i,\ j = 1, 2. \quad (55)$$

Firm i maximizes the discounted flow of its profits:

$$J_i \equiv \int_0^\infty e^{-\rho t}\pi_i\left(\mathbf{q}\left(t\right)\right) dt,\ i,\ j = 1, 2. \quad (56)$$

under the constraint (54) imposed by the dynamics of the state variable $\kappa_i(t)$. Notice that the state variable does not enter directly the objective function.

We assume that firms behave as quantity-setters. Hence, the control variable of firm i is $q_i(t)$.[19]

Define the vectors of control and state variables, respectively, as $\mathbf{q}(t)$ and $\boldsymbol{\kappa}(t)$. Then, the Hamiltonian function of firm i is:

$$\mathcal{H}_i(\mathbf{q}(t),\boldsymbol{\kappa}(t)) = e^{-\rho t}\{q_i(t)[A - B(q_i(t)+q_j(t))] + \tag{57}$$

$$+\lambda_{ii}(t)\left[f(\kappa_i(t)) - q_i(t) - \delta\kappa_i(t)\right] + \lambda_{ij}(t)\left[f(\kappa_j(t)) - q_j(t) - \delta\kappa_j(t)\right]\}$$

where $\lambda_{ij}(t) = \mu_{ij}(t)e^{\rho t}$, and $\mu_{ij}(t)$ is the co-state variable associated to $\kappa_j(t)$, i, $j = 1,2$; $j \neq i$.

The FOC on firm i's control is:

$$\frac{\partial \mathcal{H}_i(\mathbf{q}^*(t),\boldsymbol{\kappa}^*(t))}{\partial q_i} = A - 2Bq_i^*(t) - q_j(t) - \lambda_{ii}(t) = 0; \tag{58}$$

which immediately proves two facts. The first is that the instantaneous best reply function

$$q_i^*(t) = \frac{A - q_j(t) - \lambda_{ii}(t)}{2B} \tag{59}$$

is independent of state variables. The second is that, since (58) can be solved to yield the expression of $\lambda_{ii}(t)$:

$$\lambda_{ii}(t) = A - 2Bq_i^*(t) - q_j(t) \tag{60}$$

it appears that $\partial\lambda_{ii}(t)/\partial q_j(t) = -1$. These results can be summarised in the following:

Proposition 11 *The Ramsey oligopoly game is perfect but controllable. That is, the open-loop Nash equilibrium is strongly time consistent, while the Stackelberg open-loop equilibrium is time inconsistent.*

[19] It can be easily shown that none of the ensuing result modifies if firms behave as price-setters. See Cellini and Lambertini (1998).

The open-loop Nash game always produces a unique saddle point equilibrium, whose complete characterisation can be found in Cellini and Lambertini (1998) in a more general setup where N single-product firms operate in the market, supplying differentiated varieties of the same good.

5 The sticky price game

In this section, we examine a game which is uncontrollable, but not perfect. To this aim, we revisit the differential game with sticky prices dating back to Simaan and Takayama (1978) and further investigated by Fershtman and Kamien (1987) and Cellini and Lambertini (2004), *inter alia*. As in the previous section, two single-product firms, labelled as 1 and 2, offer homogeneous products over $t \in [0, \infty)$. At any time t, the inverse demand function is (53). Output levels are the control variables, while prices are the state variables, each following the dynamic equation:

$$\frac{dp(t)}{dt} \equiv \dot{p}(t) = s\left\{\widehat{p}(t) - p(t)\right\}, \; i = 1, 2, \tag{61}$$

where $\widehat{p}(t)$ denotes the notional level of the market price at time t, while $p(t)$ denotes its current level, the former being defined by the demand function (53). Notice that the speed of adjustment is captured by parameter s, with $s \in [0, \infty)$. The lower is s, the higher is the degree of price stickiness.

The instantaneous production cost function of firm i is assumed to be quadratic:

$$C_i(t) = cq_i(t) - \frac{1}{2}\left[q_i(t)\right]^2, 0 < c < A. \tag{62}$$

As a consequence, the instantaneous profit function of firm i is:

$$\pi_i(\mathbf{q}(t)) \equiv p(t)q_i(t) - C_i(t) = q_i(t) \cdot \left[p(t) - c - \frac{1}{2}q_i(t)\right], \; i = 1, 2. \qquad (63)$$

The current price of any good is generally different from its notional level. The production decisions of firms affect notional prices, but current prices evolve subject to price stickiness. We assume that firms choose the quantity to be produced, so that we are in a Cournot framework. More precisely, each firm chooses the path of her control variable $q_i(t)$ over time in order to maximize the present value of the profit flow, subject to (i) the motion laws regarding the state variables, and (ii) the initial conditions. The problem of player $i = 1, 2$ may be written as follows:

$$\begin{aligned} \max_{q_i(t)} J_i &\equiv \int_0^\infty e^{-\rho t} q_i(t) \cdot \left[p(t) - c - \frac{1}{2}q_i(t)\right] dt \\ s.t. &: \frac{dp(t)}{dt} = s\left\{\widehat{p}(t) - p(t)\right\} \end{aligned} \qquad (64)$$

and $p(0) = p_0$, $i = 1, 2$. The factor $e^{-\rho t}$ discounts future gains, and the discount rate ρ is assumed to be constant over time and equal across firms. Define the vector of controls as $\mathbf{q}(t)$. The Hamiltonian function is:

$$\begin{aligned} \mathcal{H}_i(\mathbf{q}(t), p(t)) &= e^{-\rho t}\{q_i(t)\left[p(t) - c - q_i(t)/2\right] + \\ &\quad + \lambda_i(t)s\left[A - B\left(q_i(t) + q_j(t)\right) - p_i(t)\right]\} \end{aligned} \qquad (65)$$

where $\lambda_i(t) = \mu_i(t)e^{\rho t}$, and $\mu_i(t)$ is the co-state variable associated by player i to the price $p(t)$.

The FOC taken w.r.t. the control of firm i is:

$$\frac{\partial \mathcal{H}_i(\mathbf{q}^*(t), p^*(t))}{\partial q_i} = 0 \Leftrightarrow q_i^*(t) = p^*(t) - c - \lambda_i(t)sB \qquad (66)$$

or

$$\lambda_i(t) = \frac{p^*(t) - q_i^*(t) - c}{sB} \tag{67}$$

While (66) proves that $\partial q_i^*(t)/\partial p(t) \neq 0$, $i = 1, 2$, expression (67) shows that the co-state variable of each firm is independent of the control variable of the rival. Therefore, we may state:

Proposition 12 *The sticky price oligopoly game is uncontrollable but not perfect. That is, the open-loop Nash equilibrium is only weakly time consistent, while the Stackelberg open-loop equilibrium is time consistent but not subgame perfect.*

The Nash and Stackelberg open-loop equilibria of this game are unique. For the complete characterisation as well as the stability analysis of the Nash equilibrium, we refer the reader to the aforementioned contributions.

6 Extensions and concluding remarks

In this paper we have analysed the time consistency property of open-loop equilibria, in the case of Nash and Stackelberg behaviour. We have noted that classes of games exist, in which the strong time-consistency of the open-loop Nash equilibrium associates with the time consistency of the open-loop Stackelberg equilibrium. We have labelled these setups as *perfect uncontrollable games*. We have also provided one example based on a model of oligopolistic competition with advertising efforts analysed by Leitmann and Schmitendorf (1978) and Feichtinger (1983). We have completed the exposition by briefly examining two cases where the two properties alternatively hold.

In the above analysis, we have confined our attention to two-player games, and the generalisation to the case of N players is desirable. With respect to open-loop Nash behaviour, this extension is intuitive, as state redundancy requires that property (11) is met for each player w.r.t. $N - 1$ opponents. Indeed, it can be easily shown that in the case of N players our results go through unchanged in the three models investigated in the paper. Generalising the analysis of the stackelberg case to N players is less straightforward. However, it can be easily shown that our analysis is robust in the case where there is one leader followed by $N - 1$ rivals, for the same reason as in the Nash game.

Another appealing extension would consist in investigating macroeconomic policy games where uncontrollability couples with state redundancy to ensure that optimal fiscal and monetary policies are subgame perfect. Two such examples are Xie (1997; see also Karp and Lee, 2003) and Cellini and Lambertini (2003).

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