Capacity Accumulation and Utilization in a Differential Duopoly Game[1]

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Abstract

We present a differential duopoly game with capacity accumulation, where firms control investment efforts and sales, which can be at most equal to the respective installed capacities at any point in time. We use, alternatively, inverse and direct demand functions with product differentiation, recalling Cournot and Bertrand competition. We show that, at the subgame perfect steady state equilibria, Cournot and Bertrand profits do not coincide, unless the game is quasi-static, which happens if capacity does not depreciate over time.

Keywords: differential games, capacity accumulation, price setting, quantity setting.

JEL Classification: C73, D43, D92, L13.

1 Introduction

We present a differential duopoly game with differentiated goods and capital accumulation. In particular, we model the accumulation process through reversible investment *à la* Nerlove-Arrow (1962), where capital (or capacity) accumulation occurs through costly investment, as in Solow's (1956) growth model. Firms control investment efforts and sales, which can be at most equal to the respective installed capacities at any point in time. We investigate the game using, alternatively, inverse and direct demand functions. These two settings recall the Cournot and Bertrand models, although being not equivalent, strictly speaking, to either the first or the second. However, to ease the exposition, we shall refer to 'Cournot competition' (respectively, 'Bertrand competition') when using inverse (resp., direct) demand functions.

The interest of this approach lies in the fact that it allows to construct a dynamic perspective where one can analyse the interaction between capacity and the market performance of firms, an issue that has received a considerable amout of attention in the IO literature. The main result obtained by this stream of contributions is that Bertrand competition replicates the Cournot equilibrium outcome under endogenously determined capacity constraints (Kreps and Scheinkman, 1983).

Differently from the existing models dealing with the choice of capacity in static two-stage games (see, e.g., Beckman, 1967; Levitan and Shubik, 1972; Kreps and Scheinkman, 1983; Osborne and Pitchik, 1986),[1] capital is not a control variable in the present setting, where each firm chooses its investment efforts. Consequently, the capital (or capacity) of each firm is not directly affected by the capital of opponents.

Our main results are the following. We prove that the open-loop equilibrium outcome of the Bertrand game coincides with the Cournot equilibrium. This amounts to saying that the Nerlove-Arrow-Solow differential game encompasses the results of the static analysis carried out by Kreps and Scheinkman (1983) in the spirit of the original idea dating back to Edgeworth (1897). This is technically due to the fact that the Bertrand setting obtains through a linear symmetric transformation of the Cournot setting. Therefore, the two models are isomorphic and produce the same steady state equilibrium. This, however, holds only under the open-loop solution, which

[1] As is well known, the first analysis of capacity-constrained price competition is in Edgeworth (1897).

is strongly time consistent (or subgame perfect) only if firms use inverse demand functions (i.e., in the Bertrand game). In the Cournot game, the open-loop equilibrium is not subgame perfect, and this prompts for the adoption of the closed-loop solution concept. Comparing the Cournot closed-loop equilibrium profits against the Bertrand profits (or the Cournot open-loop profits), we find that (i) they do not coincide, and (ii) there exists an admissible range of parameters (with firms selling substitute goods) where the first are smaller than the second. A situation where equilibrium profits are the same irrespective of the formulation of market demand and the choice of the solution concept is that where capacity does not depreciate at all over time. However, this case is inherently quasi-static. Therefore, we can conclude that, in a dynamic game, the coincidence between Bertrand and Cournot outcomes cannot arise in general. Moreover, the established result that Cournot behaviour is more (less) profitable than Bertrand behaviour when goods are substitutes (complements), dating back to Singh and Vives (1984), *inter alia*, is also contradicted. These results essentially depend on the fact that, at the Bertrand sunbgame perfect equilibrium firms operate at full capacity, while they keep some idle capacity at the subgame perfect Cournot equilibrium.

The remainder of the paper is structured as follows. Section 2 motivates the dynamic approach. The basic setup is laid out in section 3. The Cournot and Bertrand settings are investigated in sections 4 and 5, respectively. Then, the comparative assessment of equilibrium profits is carried out in section 6. Concluding remarks are in section 7.

2 The differential game approach

We believe that, in order to analyse the firms' behaviour concerning the accumulation of productive capacity, the differential game approach is particularly appropriate. Indeed, building capacity needs time; the depreciation of capacity over time is an important factor; the time dimension is important in planning the investment efforts towards capacity accumulation. Furhtermore, strategic interaction among firms is an important issue both in the market phase, and in the phase when capacity decisions are taken. Last but not least, the differential game approach leads to results encompassing those obtained by the more traditional approach based on static two-stage games. More precisely, the differential game approach allows to understand *under* *which particular conditions*, the results from static game can be interpreted as the steady state solution of the differential game.[2] Specifically, in the present paper we aim at checking whether the well-known result obtained by Kreps and Scheinkman (1983) about the equivalence between the Cournot and the Bertrand setting in a two-stage static game, holds in a differential game framework too.

The existing literature on differential games mainly focusses on two kinds of strategies adopted by players: the open-loop and the closed-loop strategies.[3] When players adopt the open-loop solution concept, they design the time path concerning the control variable(s) at the initial time and then stick to it forever. This means that the open-loop strategy is simply a time path of actions, and time is the only determinant of the action to be done at any instant. The relevant equilibrium concept is the open-loop Nash equilibrium, which is only weakly time consistent and therefore, in general, it is not subgame perfect.[4] When players adopt the closed-loop strategy, they do not precommit control variable(s) on any path, and their actions at any instant may depend on the history of the game up to that instant, and, in particular, on the values of the state variables. In this situation, the information set used by players in setting their actions at any given time is often simplified to be only the current value of the state variables at that time, along with the initial conditions. This specific situation is labelled as *memoryless closed-loop* (Mehlmann, 1988). The relevant equilibrium concept, in this case, is the closed-loop memoryless Nash equilibrium, which is strongly time consistent (or subgame perfect).[5]

The literature on differential games devotes a considerable amount of

[2] For additional motivation supporting the choice of the differential game approach, see Dockner *et al.* (2000), chapter 9. This reference also provides a presentation of relevant differential games with capital accumulation.

[3] See Kamien and Schwartz (1981); Basar and Olsder (1982, 1995^2); Mehlmann (1988); Dockner *et al.* (2000). See also Cellini and Lambertini (2001).

[4] As to the definition of time consistency and subgame perfection, we rely - among many different definitions available in the literature - on the definition provided by Dockner *et al.* (2000, Section 4.3). We use "strong time consistency" as a synonomous of subgame perfection.

[5] Different rules of closed-loop class do exist: e.g., the perfect-memory closed loop, where the control variables depend on the complete history of the state variable(s); the feedback rule, where only the current stock of states are considered, irrespective of initial conditions. For oligopoly models where firms follow feedback rules, see Simaan and Takayama (1978), Fershtman and Kamien (1987, 1990), Dockner and Haug (1990), *inter alia*.

attention to identifying classes of games where the closed-loop equilibria degenerate into open-loop equilibria. The degeneration means that the Nash equilibrium time paths of control variables coincide under the different strategy concepts. Whenever an open-loop equilibrium is a degenerate closed-loop equilibrium, then the former is also strongly time consistent (or subgame perfect). Therefore, one can rely upon the open-loop equilibrium which, in general, is much easier to derive than closed-loop.[6]

3 The model

The game is played over continuous time, $t \in [0, \infty)$.[7] In any instant of time, the market is served by two firms, 1 and 2, producing a differentiated good. Let $q_i(t)$ define the quantity sold by firm $i = 1, 2$ at time t. The marginal production cost is constant and equal to c for all firms. For the sake of simplicity, we pose $c = 0$. As in Singh and Vives (1984), the demand function for good i at time t is:

$$p_i(t) = A - q_i(t) - Dq_j(t) \ , (j \neq i) \tag{1}$$

where parameter $D \in [-1, 1]$ measures the degree of substitutability or complementarity between goods. If $D = 1$, they are perfect substitutes; if $D = 0$ the goods are independent and each firm behaves as a monopolist; if $D = -1$, products are perfect complements.

In order to produce, firms must accumulate capacity or physical capital $k_i(t)$ over time. As in Solow (1956) and Nerlove and Arrow (1962), the relevant dynamic equation describing the accumulation of capacity (or physical capital) is:[8]

$$\frac{\partial k_i(t)}{\partial t} = I_i(t) - \delta k_i(t) \ , \tag{2}$$

[6] Classes of games where this coincidence arises are illustrated in Clemhout and Wan (1974); Reinganum (1982); Mehlmann and Willing (1983); Dockner *et al.* (1985); Fershtman (1987); Fershtman *et al.* (1992). For an overview, see also Mehlmann (1988), and Dockner *et al.* (2000, ch. 7).

[7] The game can be reformulated in discrete time without significantly affecting its qualitative properties. For further details, see Basar and Olsder (1982, 1995^2).

[8] A similar setting is used in Fudenberg and Tirole (1983), Fershtman and Muller (1984) and Reynolds (1987). For a dynamic oligopoly game where capital accumulates *à la* Ramsey (i.e., current unsold output becomes additional productive capacity), see Cellini and Lambertini (1998).

where $I_i(t)$ is the investment carried out by firm i at time t, and $\delta > 0$ is the constant depreciation rate. The instantaneous cost of investment is $C_i[I_i(t)] = b[I_i(t)]^2$, with $b > 0$.

We also assume that firms bear instantaneous production costs $C_i(t) = cq_i(t)$ and, for the sake of simplicity, we set $c = 0$ without further loss of generality. At any time t, firm i produces and sells an output $q_i(t) \leq k_i(t)$, so that we can write $q_i(t) = \alpha_i(t) k_i(t)$, with $\alpha_i(t) \in (0, 1]$. As a consequence, the demand function for variety i rewrites as:

$$p_i(t) = A - \alpha_i(t) k_i(t) + D\alpha_j(t) k_j(t). \tag{3}$$

Player i's objective is the maximisation of the present value of the profit flows:

$$\max_{I_i} \int_0^\infty \pi_i(., t) e^{-\rho t} dt \tag{4}$$

subject to the dynamic constraint represented by the behaviour of the state variables (2) for $i = 1, 2$. Instantaneous profit is $\pi_i(., t) = p_i q_i - b[I_i(t)]^2$. The factor $e^{-\rho t}$ discounts future gains, and the discount rate ρ is assumed to be constant and common to both players. In order to solve his optimisation problem, each player defines a strategy.

The control variables of firm i are the instantaneous investment $I_i(t)$ and the capacity utilization level $\alpha_i(t)$, while $k_i(t)$ is obviously a state variable.

4 Cournot competition

When firms write profits using the inverse demand functions (3), the Hamiltonian function of firm i writes as follows:

$$\begin{aligned} \mathcal{H}_i(t) &= e^{-\rho t} \cdot \left\{ [A - \alpha_i(t) k_i(t) - D\alpha_j(t) k_j(t)] \alpha_i(t) k_i(t) - b[I_i(t)]^2 + \right. \\ & \quad \left. \lambda_{ii}(t)[I_i(t) - \delta k_i(t)] + \lambda_{ij}(t)[I_j(t) - \delta k_j(t)] \right\} \end{aligned} \tag{5}$$

where $\lambda_{ij}(t) = \mu_{ij}(t) e^{\rho t}$, and $\mu_{ij}(t)$ is the co-state variable associated to $k_j(t)$, $i, j = 1, 2$. Moreover, let $k_i(0) \equiv k_{i0}$ define the initial condition for firm i.

4.1 The open-loop Nash equilibrium

Examine the open-loop simultaneous solution. The first order conditions (FOCs) are (we omit exponential discounting for brevity):

$$\frac{\partial \mathcal{H}_i(t)}{\partial \alpha_i(t)} = k_i(t)[A - 2\alpha_i(t) k_i(t) - D\alpha_j(t) k_j(t)] = 0 \tag{6}$$

$$\frac{\partial \mathcal{H}_i(t)}{\partial I_i(t)} = -2bI_i(t) + \lambda_{ii}(t) = 0 \tag{7}$$

$$-\frac{\partial \mathcal{H}_i(t)}{\partial k_i(t)} = \frac{\partial \lambda_{ii}(t)}{\partial t} - \rho\lambda_{ii}(t) \Rightarrow$$

$$\frac{\partial \lambda_{ii}(t)}{\partial t} = (\rho + \delta)\,\lambda_{ii}(t) - \alpha_i\,(t)\,[A - 2\alpha_i\,(t)\,k_i(t) - D\alpha_j\,(t)\,k_j(t)] \tag{8}$$

together with initial conditions $k_i\,(0) > 0$ and transversality conditions:

$$\lim_{t\to\infty} \mu_{ii}(t)\cdot k_i(t) = 0\ ;\ \lim_{t\to\infty} \mu_{ij}(t)\cdot k_j(t) = 0\ . \tag{9}$$

From (7) we obtain:

$$\lambda_{ii}(t) = 2bI_i(t)\,;\ \frac{\partial I_i(t)}{\partial t} = \frac{1}{2b}\frac{\partial \lambda_{ii}(t)}{\partial t} \tag{10}$$

Then, using the symmetry conditions $\alpha_i\,(t) = \alpha_j\,(t) = \alpha\,(t)$ and $k_i\,(t) = k_j\,(t) = k\,(t)$ and $\lambda_{ii}(t) = \lambda_{jj}(t) = 2bI(t)$, we can write (the indication of time is omitted for brevity):

$$\frac{\partial I}{\partial t} = \frac{2b\,(\rho + \delta)\,I - \alpha\,[A - \alpha\,(2 + D)\,k]}{2b} \tag{11}$$

which can be further simplified using $k^* = I/\delta$ from $\partial k/\partial t = 0$:

$$\frac{\partial I}{\partial t} = \frac{[2b\delta\,(\rho + \delta) + \alpha^2\,(2 + D)]\,I - \alpha\delta A}{2b} \tag{12}$$

The above differential equation is equal to zero at:

$$I^* = \frac{\alpha\delta A}{2b\delta\,(\rho + \delta) + \alpha^2\,(2 + D)} \tag{13}$$

This, together with $k^* = I^*/\delta$, can be plugged into (6), that rewrites as follows:

$$\frac{\partial \mathcal{H}}{\partial \alpha} = \frac{2A^2\alpha\delta b\,(\rho + \delta)}{[2b\delta\,(\rho + \delta) + \alpha^2\,(2 + D)]} > 0 \text{ always.} \tag{14}$$

Therefore, $\alpha^{ss} = 1$, with the superscript ss standing for *steady state*. The corresponding optimal steady state levels of investment and capacity are:

$$I^{ss} = \frac{\delta A}{2 + D + 2b\,(\rho + \delta)\,\delta}\ ;\ k^{ss} = \frac{I^{ss}}{\delta} = \frac{A}{2 + D + 2b\,(\rho + \delta)\,\delta}\ . \tag{15}$$

The pair $\{I^{ss}, k^{ss}\}$ is a saddle point; the proof is straightforward, in that, the determinant of the Jacobian matrix associated to the dynamic system $\{\partial k_i(t)/\partial t = 0, \partial I_i(t)/\partial t = 0\}$ is negative, while its trace is positive.[9]

[9] Calculations are trivial and they are omitted for brevity.

Moreover, from (15) it can be shown that, in $\delta = 0$:

$$k^{ss} = q^{ss} = \frac{A}{2+D} \tag{16}$$

which coincides with the equilibrium output of the static game studied by Singh and Vives (1984). In general, however:

Proposition 1 *For all positive and admissible values of parameters $\{\delta, \rho\}$, steady state (open-loop Nash equilibrium) capacity and sales are lower than in the static Cournot game with product differentiation.*

Steady state profits are:

$$\pi^{ss}_{ol} = \frac{A^2 \left[1 + b\delta \left(2\rho + \delta\right)\right]}{\left\{2\left[1 + b\delta\left(\rho + \delta\right)\right] + D\right\}^2} \tag{17}$$

where the subscript ol stands for *open-loop*. Profits π^{ss}_{ol} coincide with the Cournot-Nash equilibrium profits $\pi^{CN} = A^2/(2+D)^2$ attained in the static model when $\delta = 0$. In general, the effects of parameters $\{\delta, \rho\}$ on steady state profits are described by the following partial derivatives:

$$\frac{\partial \pi^{ss}_{ol}}{\partial \delta} = -\frac{2A^2 b \left[2b\delta^2 \left(3\rho + \delta\right) + 2\delta\left(1 + b\rho^2\right) - D\left(\rho + \delta\right)\right]}{\left\{2\left[1 + b\delta\left(\rho + \delta\right)\right] + D\right\}^3} ; \tag{18}$$

$$\frac{\partial \pi^{ss}_{ol}}{\partial \rho} = -\frac{2A^2 b\delta \left(2b\delta\rho - D\right)}{\left\{2\left[1 + b\delta\left(\rho + \delta\right)\right] + D\right\}^3} . \tag{19}$$

This simple comparative static exercise suffices to prove the following:

Proposition 2 *The steady state (open-loop Nash equilibrium) profits are non-monotone in both ρ and δ.*

The above Proposition, in turn, implies that there exist parameter ranges wherein the steady state profits π^{ss} generated by the differential game are larger than the static equilibrium profits π^{CN}:

$$\pi^{ss}_{ol} - \pi^{CN} \propto (2 + D)\left[\delta\left(D - 2\right) + 2D\rho\right] - 4b\delta\left(\rho + \delta\right)^2 , \tag{20}$$

with the r.h.s. of (20) being equal to zero at:

$$D = \frac{-2\rho \pm 2\left(\rho + \delta\right)\sqrt{1 + b\delta\left(\delta + 2\rho\right)}}{\delta + 2\rho} . \tag{21}$$

The smaller root is negative, while the larger root is positive and smaller than one for all

$$b < \frac{3(2\rho - \delta)}{4\delta(\delta + \rho)^2} \tag{22}$$

provided that $\rho > \delta/2$. This proves the following Corollary to Proposition 2:

Corollary 3 *Suppose $\rho > \delta/2$. If so, then $\pi_{ol}^{ss} > \pi^{CN}$ for all*

$$D \in \left(\frac{-2\rho + 2(\rho + \delta)\sqrt{1 + b\delta(\delta + 2\rho)}}{\delta + 2\rho}, 1 \right].$$

4.2 The closed-loop Nash equilibrium

Examine the closed-loop simultaneous solution. FOCs (6-7), initial conditions and transversality conditions are the same as above, while the costate equations now write as follows:

$$-\frac{\partial \mathcal{H}_i(t)}{\partial k_i(t)} - \frac{\partial \mathcal{H}_i(t)}{\partial \alpha_j(t)} \cdot \frac{\partial \alpha_j^*(t)}{\partial k_i(t)} - \frac{\partial \mathcal{H}_i(t)}{\partial I_j(t)} \cdot \frac{\partial I_j^*(t)}{\partial k_i(t)} = \frac{\partial \lambda_{ii}(t)}{\partial t} - \rho \lambda_{ii}(t) \tag{23}$$

where the terms

$$\frac{\partial \mathcal{H}_i(t)}{\partial \alpha_j(t)} \cdot \frac{\partial \alpha_j^*(t)}{\partial k_i(t)} ; \frac{\partial \mathcal{H}_i(t)}{\partial I_j(t)} \cdot \frac{\partial I_j^*(t)}{\partial k_i(t)} \tag{24}$$

describe the feedback effects from the rival's control variables, and starred variables are defined as the solutions to the respective FOCs. First of all, note that the feedback from the rival's investment is nil, since:

$$\frac{\partial I_j^*(t)}{\partial k_i(t)} = 0 \tag{25}$$

as it can be immediately ascertained from (7). Second, from (6) we obtain:

$$\alpha_j^*(t) = \frac{A - D\alpha_i(t) k_i(t)}{2k_j(t)} \Rightarrow \frac{\partial \alpha_j^*(t)}{\partial k_i(t)} = -\frac{D\alpha_i(t)}{2k_j(t)} \tag{26}$$

while

$$\frac{\partial \mathcal{H}_i(t)}{\partial \alpha_j(t)} = -D\alpha_i(t) k_i(t) k_j(t). \tag{27}$$

We may rewrite the costate equation (23) accordingly:

$$\frac{\partial \lambda_{ii}(t)}{\partial t}=\frac{2\left(\rho+\delta\right)\lambda_{ii}(t)-\alpha_i\left(t\right)\left[2A-\alpha_i\left(t\right)k_i(t)\left(4-D^2\right)-2D\alpha_j\left(t\right)k_j(t)\right]}{2} \tag{28}$$

Then, using

$$\lambda_{ii}(t)=2bI_i(t)\,;\;\frac{\partial I_i(t)}{\partial t}=\frac{1}{2b}\frac{\partial \lambda_{ii}(t)}{\partial t} \tag{29}$$

and imposing the same symmetry cionditions as above, we obtain the differential equation of firm i's investments:

$$\frac{\partial I}{\partial t}=\frac{4b\left(\rho+\delta\right)I-\alpha\left[2A-\alpha\left(4+D\left(2-D\right)\right)k\right]}{4b}\,. \tag{30}$$

Solving the system

$$\left\{\frac{\partial \mathcal{H}}{\partial \alpha}=0;\frac{\partial I}{\partial t}=0;\frac{\partial k}{\partial t}=0\right\} \tag{31}$$

we characterise the steady state equilibrium of the closed-loop Cournot-Nash game:

$$\alpha^{ss}\;=\;\frac{2\sqrt{b\delta\left(\rho+\delta\right)}}{D}\,;\;I^{ss}=\frac{AD\delta}{2\left(2+D\right)\sqrt{b\delta\left(\rho+\delta\right)}}\,; \tag{32}$$

$$k^{ss}\;=\;\frac{I^{ss}}{\delta}=\frac{AD}{2\left(2+D\right)\sqrt{b\delta\left(\rho+\delta\right)}} \tag{33}$$

as long as

$$\alpha^{ss}\leq 1\Rightarrow b\leq\widehat{b}\equiv\frac{D^2}{4\delta\left(\rho+\delta\right)}\,. \tag{34}$$

The corresponding equilibrium profits are:

$$\pi_{cl}^{ss}=\frac{A^2\left[4\left(\rho+\delta\right)-\delta D^2\right]}{4\left(\rho+\delta\right)\left(2+D\right)^2}>0\text{ iff }D<2\sqrt{\frac{\rho+\delta}{\delta}}\,, \tag{35}$$

where subscript cl stands for *closed-loop*. This condition is always met, since

$$2\sqrt{\frac{\rho+\delta}{\delta}}>1 \tag{36}$$

for all ρ and δ. Moreover, notice that

$$\lim_{\rho\to\infty}\pi_{cl}^{ss}=\lim_{\delta\to 0}\pi_{cl}^{ss}=\frac{A^2}{\left(2+D\right)^2}=\pi^{CN} \tag{37}$$

while, in general, $\pi_{cl}^{ss} < \pi^{CN}$ for all finite values of ρ and all $\delta \in (0, 1]$. If $\delta = 0$, then $\pi_{cl}^{ss} = \pi^{CN}$.

Now examine the stability properties of the dynamic system (2-30), on the basis of the associate Jacobian matrix:

$$J = \begin{bmatrix} -\delta & 1 \\ \dfrac{\alpha^2 \left[4 + D(2 - D)\right]}{4b} & \rho + \delta \end{bmatrix} \tag{38}$$

The trace and determinant of matrix J are $T(J) = \rho > 0$ and:

$$\Delta(J) = -\delta(\rho + \delta) - \frac{\alpha^2 \left[4 + D(2 - D)\right]}{4b} \tag{39}$$

which is always negative. Therefore, the closed-loop equilibrium is a saddle point in the whole parameter range. The foregoing analysis can be summarised by:

Proposition 4 *The closed-loop Cournot game yields a unique subgame perfect equilibrium, which is always a saddle point. Steady state profits π^{ss} are lower than the static Cournot-Nash profits π^{CN} except when, in the limit, either δ tends to zero or ρ tends to infinity (or both). In such cases, $\pi_{cl}^{ss} = \pi^{CN}$. If the cost of investment is low enough, firms hold excess capacity in equilibrium.*

The last part of the Proposition deserves a comment. The presence of idle capacity at the subgame perfect equilibrium is a direct consequence of the fact that, in the closed-loop game, each firm explicitly takes into account the rival's reaction. In closed-loop oligopoly games, it is commonly observed that firms sell more than in open-loop games, *all else equal* (see, e.g., Fershtman and Kamien, 1987). This is due to the fact that each firm tries to anticipate the rival's behaviour and, in so doing, to acquire a larger market share. Here, the decisions on capacity and sales are separated, and the attempt at outselling one's rival translates into the holding of some idle capacity, as long as this is sufficiently inexpensive.

The above Proposition has an interesting Corollary:

Corollary 5 *At the closed-loop equilibrium, firms' sales are equal to the static Cournot-Nash output. Hence, the steady state price at the closed-loop equilibrium corresponds to the static Cournot-Nash price.*

Proof. To prove this result, it suffices to observe from (32) that the product of α^{ss} and k^{ss} corresponds to $\alpha^{ss}k^{ss} = A/(2+D) = q^{CN}$. Accordingly, $p^{ss} = A/(2+D) = p^{CN}$. ■

Consequently, the result $\pi_{cl}^{ss} < \pi^{CN}$ is entirely determined by the presence of the instantaneous investment costs involved by capacity accumulation.

5 Bertrand competition

Now examine price competition. Inverting the demand system (3), we obtain:

$$k_i(t) = \frac{A(1-D) - p_i(t) + Dp_j(t)}{\alpha_i(t)(1-D^2)} \tag{40}$$

yielding the following Hamiltonian function of firm i :

$$\begin{aligned}\mathcal{H}_i(t) = e^{-\rho t} \cdot \Bigg\{ & \frac{p_i(t)\left[A(1-D) - p_i(t) + Dp_j(t)\right]}{(1-D^2)} - b\left[I_i(t)\right]^2 + \\ & + \lambda_{ii}(t)\left[I_i(t) - \frac{\delta\left(A(1-D) - p_i(t) + Dp_j(t)\right)}{\alpha_i(t)(1-D^2)}\right] + \\ & + \lambda_{ij}(t)\left[I_j(t) - \frac{\delta\left(A(1-D) - p_j(t) + Dp_i(t)\right)}{\alpha_j(t)(1-D^2)}\right]\Bigg\}\end{aligned} \tag{41}$$

where $\lambda_{ij}(t) = \mu_{ij}(t)e^{\rho t}$, and $\mu_{ij}(t)$ is the co-state variable associated to $k_j(t)$, $i, j = 1, 2$. However, in the present setting, the state equations are defined in terms of prices rather than capacities, with prices being auxiliary state variables. Therefore, let $p_i(0) \equiv p_{i0}$ define the initial condition for firm i.

Observe the FOCs on the control variables (again, the indication of time is omitted henceforth):

$$\frac{\partial \mathcal{H}_i}{\partial \alpha_i} = \frac{\delta\lambda_{ii}\left[A(1-D) - p_i + Dp_j\right]}{\alpha_i^2(1-D^2)} = \frac{\delta\lambda_{ii}k_i}{\alpha_i} = 0 \tag{42}$$

$$\frac{\partial \mathcal{H}_i}{\partial I_i} = -2bI_i + \lambda_{ii} = 0 \Rightarrow \lambda_{ii} = 2bI_i \tag{43}$$

$$\frac{\partial I_i}{\partial t} = \frac{1}{2b} \cdot \frac{\partial \lambda_i}{\partial t} \tag{44}$$

As long as δ, α_i and k_i are positive, condition (42) can be satisfied only in $\lambda_{ii} = 0$, which in turn cannot be the case in general, unless $I_i = 0$, as it can be immediately ascertained from (43). Therefore, in general:

$$\frac{\partial \mathcal{H}_i}{\partial \alpha_i} > 0 \tag{45}$$

and it follows that $\alpha_i = \alpha_j = 1$ at any t. This also entails an additional relevant implication, i.e., we cannot obtain a best reply function describing α_i^* from (42), let alone the feedback effect $\partial \alpha_i^* / \partial p_j$. This, in combination with (43), ultimately entails:

Proposition 6 *In the Bertrand setting, firms always operate at full capacity, and the open-loop Nash equilibrium is a degenerate closed-loop equilibrium. Therefore, the open-loop solution is strongly time consistent (or subgame perfect).*

The above Proposition establishes that the present game is a 'linear state game', producing Markov perfect open-loop Nash equilibria.[10] The Nash equilibrium time path of the variables coincides under opne-loop and closed-loop decision rules. This is due to two features of the game: first, the dynamic behaviour of any firm's state variable (i.e., capacity) does not depend on the rivals' control and state variables, which makes the kinematic equations concerning other firms redundant; second, for any firm, the first order conditions taken w.r.t. the control variables are independent of the rivals' state variables, which entails that the cross effect from rivals' states to own controls (which characterises the closed-loop information structure) disappears.

Notice also that, on the basis of the above considerations, the costate variable λ_{ij} is redundant in that it does not appear in the first order conditions (42) and (43).[11] Therefore, we can simplify the problem, by setting $\alpha_i = \alpha_j = 1$. Moreover, $\lambda_{ij} = 0$ and $\lambda_{ii} = \lambda_i$, as only one co-state variable is relevant for any player. Accordingly, we can derive the open-loop costate equation. Since we are using direct demand functions, capacity k_i is expressed as a function of the price vector $\{p_i\,,\, p_j\}$. Therefore,

$$\frac{\partial \mathcal{H}_i}{\partial k_i} = \frac{\partial \mathcal{H}_i}{\partial p_i} \cdot \frac{\partial p_i}{\partial k_i} + \frac{\partial \mathcal{H}_i}{\partial p_j} \cdot \frac{\partial p_j}{\partial k_i} \tag{46}$$

[10] A fixed price version of the game is in Leitmann and Schmitendorf (1978) and Feichtinger (1983). In their contributions, the setup is interpreted as a game of advertising. For a thorough exposition of linear state games, see Dockner *et al.* (2000, chapter 7).

[11] This is specifically due to the fact that the state equations are separated.

where

$$\frac{\partial \mathcal{H}_i}{\partial p_i} = \frac{A(1-D) - 2p_i + Dp_j + \delta\lambda_i}{1-D^2} ;$$
$$\frac{\partial \mathcal{H}_i}{\partial p_j} = \frac{D\left[p_i - \delta\lambda_i\right]}{1-D^2} ; \tag{47}$$
$$\frac{\partial p_i}{\partial k_i} = -1 ; \; \frac{\partial p_j}{\partial k_i} = -D .$$

Partial derivatives $\partial p_i/\partial k_i$ and $\partial p_i/\partial k_j$ are calculated using the inverse demand function (1). Using (47), the costate equation writes as follows:

$$-\left[\frac{\partial \mathcal{H}_i}{\partial p_i}\frac{\partial p_i}{\partial k_i} + \frac{\partial \mathcal{H}_i}{\partial p_j}\frac{\partial p_j}{\partial k_i}\right] = \frac{\partial \lambda_i}{\partial t} - \rho\lambda_i \tag{48}$$

from which we obtain

$$\frac{\partial \lambda_i}{\partial t} = \frac{A\left(1-D\right) - p_i\left(2-D^2\right) + Dp_j + \lambda_i\left(\rho+\delta\right)\left(1-D^2\right)}{1-D^2} . \tag{49}$$

Then, plugging (43) and (49) into (44) and imposing the symmetry condition $p_j = p_i$, we have:

$$\frac{\partial I_i}{\partial t} = \frac{A - p_i\left(2+D\right) + 2bI_i\left(\rho+\delta\right)\left(1+D\right)}{2b\left(1+D\right)} \tag{50}$$

with

$$\frac{\partial I_i}{\partial t} = 0 \text{ at } I_i^{ss} = \frac{p_i\left(2+D\right) - A}{2b\left(\rho+\delta\right)\left(1+D\right)} . \tag{51}$$

I_i^{ss} can be substituted into (2), which simplifies as follows:

$$\frac{\partial k_i}{\partial t} = \frac{p_i\left(2+D\right) - A}{2b\left(\rho+\delta\right)\left(1+D\right)} - \frac{\delta\left[A - p_i\right]}{1+D} \tag{52}$$

with

$$\frac{\partial k_i}{\partial t} = 0 \text{ at } p_i^{ss} = \frac{A\left[1 + 2b\delta\left(\rho+\delta\right)\right]}{2\left[1 + b\delta\left(\rho+\delta\right)\right] + D} \tag{53}$$

Now, using (53), we can simplify the expression for the steady state levels of investment and capacity:

$$I_i^{ss} = \frac{\delta A}{2 + D + 2b\left(\rho+\delta\right)\delta} ; \; k_i^{ss} = \frac{I_i^{ss}}{\delta} , \tag{54}$$

which coincide with (15). Also the equilibrium profits are obviously the same as in the Cournot game investigated in section 4.1 (see expression (17)).

This discussion leads to the following:

Proposition 7 *The steady state of the open-loop Bertrand game is observationally equivalent from that of the open-loop Cournot game.*

Note, however, that the open-loop Nash equilibrium of the Cournot game is not subgame perfect. Therefore:

Corollary 8 *At the subgame perfect equilibria, the Bertrand outcome does not coincide with the Cournot outcome, with the same capacity accumulation dynamics being used in both settings.*

6 Closed-loop vs open-loop equilibria

In the previous section, we have established that, irrespective of whether one uses direct or inverse demand functions, the open-loop equilibrium looks exactly the same. However, from sections 4.1 and 4.2 we know that (i) the open-loop equilibrium of the Cournot model is not subgame perfect, and (ii) at the closed-loop Cournot equilibrium, in general, firms will not operate at full capacity. This prompts for a comparative assessment of the performance of firms under the two alternative solution concepts.

We confine our attention to the case where the closed-loop Cournot game yields an interion solution w.r.t. α^{ss}, i.e., $b \leq \widehat{b} \equiv D^2 / [4\delta(\rho+\delta)]$. Evaluating (17) against (35), we simultaneously compare

- the Bertrand profits *versus* the Cournot profits, at the subgame perfect equilibria of the respective games;
- the open-loop Cournot profits *versus* the closed-loop Cournot profits.

In games where the size of the firm coincides with its sales, the answer would be straightforward, since the closed-loop solution usually entails larger sales and lower profits as compared to the open-loop solution (see, e.g., Fershtman and Kamien, 1987). Here, however, firm size (as measured by its installed capacity in steady state) may differ from its sales. Accordingly, this assessment must be explicitly carried out on the basis of:

$$\text{sign}\{\pi_{ol}^{ss} - \pi_{cl}^{ss}\} = \text{sign}\left\{-4b^2\delta(\rho+\delta)\left[4(\rho+\delta) - \delta D^2\right] + \right. \tag{55}$$
$$\left. 4b(\rho+\delta)(2+D)[2\rho D - \delta(1-D)(2+D)] + D^2(2+D)^2\right\}.$$

The r.h.s. of (55) may take either sign depending upon the size and sign of the four parameters $\{b, D, \rho, \delta\}$. Taking the roots of $\pi_{ol}^{ss} - \pi_{cl}^{ss} = 0$ w.r.t. b, we obtain:

$$b_1 = \frac{(2+D)\left[2\rho D - \delta(1-D)(2+D)\right] + \sqrt{\Omega}}{2\delta(\rho+\delta)\left[4(\rho+\delta) - \delta D^2\right]}$$
$$b_1 = \frac{(2+D)\left[2\rho D - \delta(1-D)(2+D)\right] - \sqrt{\Omega}}{2\delta(\rho+\delta)\left[4(\rho+\delta) - \delta D^2\right]} \tag{56}$$

where:

$$\Omega \equiv 4\rho D\left[\rho D + \delta\left(D^2 + 2D - 2\right)\right] + \delta^2(2+D)\left(2D^2 - 3D + 2\right). \tag{57}$$

Hence, two cases may arise:

- $\Omega < 0$. If so, the equation $\pi_{ol}^{ss} - \pi_{cl}^{ss} = 0$ has no real roots, entailing that $\pi_{ol}^{ss} - \pi_{cl}^{ss} < 0$ always, since the coefficient of b^2 is always negative.

- $\Omega \geq 0$. If so, $b_1, b_2 \in \mathbb{R}$, with $b_2 < 0 < \widehat{b} < b_1$ in the whole admissible range of parameters, and $\pi_{ol}^{ss} - \pi_{cl}^{ss} > 0$ for all $b \leq \widehat{b}$.

Instead of spanning the whole parameter range, consider the following example. Suppose $b = \widehat{b}/2$. In such a case,

$$\pi_{ol}^{ss} - \pi_{cl}^{ss} \propto D^2\left\{4\rho D(8+3D) + \delta(2+D)\left[D^2(6+D) + 8(2+3D)\right]\right\} \tag{58}$$

and $\pi_{ol}^{ss} > \pi_{cl}^{ss}$ iff

$$\frac{\rho}{\delta} > \max\left\{-\frac{(2+D)\left[D^2(6+D) + 8(2+3D)\right]}{4D(8+3D)}, 0\right\} \tag{59}$$

with

$$-\frac{(2+D)\left[D^2(6+D) + 8(2+3D)\right]}{4D(8+3D)} \gtrless 0 \text{ for all } D \lessgtr 0.$$

Therefore, over the whole substitutability range, $\pi_{ol}^{ss} > \pi_{cl}^{ss}$ for all admissible values of ρ and δ. If instead $D \in [-1, 0)$, then $\pi_{ol}^{ss} > \pi_{cl}^{ss}$ iff

$$\frac{\rho}{\delta} > -\frac{(2+D)\left[D^2(6+D) + 8(2+3D)\right]}{4D(8+3D)} \tag{60}$$

and conversely. If $D = 0$, obviously $\pi_{ol}^{ss} = \pi_{cl}^{ss}$.

With reference to the literature comparing the relative profitability of Bertrand and Cournot markets, the above argument proves our final result:

Proposition 9 *If goods are substitutes, there are admissible parameter constellations wherein the Bertrand game is more profitable than the Cournot game. Conversely, if goods are complements, there are admissible parameter constellations wherein the opposite holds.*

This is in sharp contrast with the acquired wisdom pertaining to static games, establishing that Cournot is more (less) profitable than Bertrand when firms sell substitute (complement) goods. The source of this result lies in the fact that, in the present differential game, capacity accumulation entails an instantaneous adjustment cost which is, by definition, absent in the corresponding static game where capacity is set up from scratch. Moreover, steady state sales differ across equilibria. In particular, provided that $\alpha_{cl}^{ss} < 1$:

$$q^{CN} - k_{ol}^{ss} = \frac{2Ab\delta(\rho+\delta)}{(2+D)[D+2(1+b\delta(\rho+\delta))]} > 0 \text{ always,} \tag{61}$$

which means that, at the closed-loop equilibrium, firms sell more than at the open-loop equilibrium, although leaving some capacity idle. Accordingly, the Cournot equilibrium price is lower than the Bertrand equilibrium price.

Finally, there remains to be stressed that, in the (rather unrealistic) case where $\delta = 0$, the three equilibria investigated above are observationally equivalent in terms of profits, for all values of D and ρ (the latter parameter being irrelevant in such a case):

Proposition 10 *If the depreciation rate of installed capacity is nil, then steady state equilibrium profits always coincide with the static Cournot-Nash profits, irrespective of whether goods are substitutes, compements, or independent.*

This Proposition emphasises that the correspondence between Cournot and Bertrand outcome when competition is essentially static obtains as a special case of a dynamic model where the profit performance of firms, in general, will substantially differ across different settings.

7 Concluding remarks

We have investigated the influence of costly capacity accumulation on equilibrium profits in a differential game where firms optimally choose their respective investment efforts as well as the degree of capacity utilization. This has allowed us to investigate two alternative settings mimicking Cournot and Bertrand behaviour, to show that, in general, the equivalence between Bertrand and Cournot outcomes under capacity constraints, holding true in static games (Kreps and Scheinkman, 1983), does not carry over to dynamic games. Moreover, we have also shown that Bertrand may be more profitable than Cournot, even when firms supply substitute goods. These results are driven by the drastically different attitude of firms towards capacity accumulation and utilization in the two settings. While they operate at full capacity at the Bertrand equilibrium, they keep some excess capacity at the Cournot equilibrium. Moreover, in the latter the installed capacity and the associated sales are larger than in the former, and the Cournot firm bears larger costs and earns lower revenues than a Bertrand firm.

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