Price vs Quantity in a Dynamic Duopoly Game with Capacity Accumulation

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September 18, 2002

Abstract

We present a differential duopoly game with capacity accumulation, where the closed-loop memoryless Nash equilibrium collapses into the open-loop Nash equilibrium. Symmetric Bertrand and Cournot equilibria are observationally equivalent. This result encompasses the conclusions of the well known model by Kreps and Scheinkman (1983) concerning the equivalence between the Bertrand and Cournot settings in a static two-stage game. However, in the dynamic framework, the equilibrium outcome is different, when one firm plays *à la* Bertrand and the other *à la* Cournot. This has relevant bearings upon firms' endogenous choice of the market variable, as well as the social desirability of prices vs quantities. For some admissible parameter ranges, the conflict between private and social incentives concerning the choice between prices and quantities disappears.

Keywords: differential games, capacity accumulation, price setting, quantity setting.

JEL Classification: C73, D43, D92, L13.

1 Introduction

We present a differential duopoly game with differentiated goods and capital accumulation. In particular, we model the accumulation process through reversible investment *à la* Nerlove-Arrow (1962), where capital (or capacity) accumulation occurs through costly investment, as in Solow's (1956) growth model. We show that the open-loop Nash equilibrium coincides with the closed-loop memoryless equilibrium, and hence it is strongly time consistent. Differently from the existing models dealing with the choice of capacity in static two-stage games (see, e.g., Beckman, 1967; Levitan and Shubik, 1972; Kreps and Scheinkman, 1983; Osborne and Pitchik, 1986), capital is not a control variable in the present setting, where each firm chooses its investment efforts. Consequently, the capital (or capacity) of each firm is not directly affected by the capital of opponents.

We prove that the equilibrium outcome of the Bertrand game coincides with the Cournot equilibrium. This amounts to saying that the Nerlove-Arrow-Solow differential game encompasses the results of the static analysis carried out by Kreps and Scheinkman (1983) in the spirit of the original idea dating back to Edgeworth (1897). This is technically due to the fact that the Bertrand setting obtains through a linear symmetric transformation of the Cournot setting. Therefore, the two models are isomorphic and produce the same steady state equilibrium.

Then, we deal with the mixed case where one firm is a price setter while the other is a quantity setter. In such a case, the transformation that must be operated to obtain the asymmetric model from either the Bertrand or the Cournot model is asymmetric. In particular, the asymmetry affects state and co-state equations in such a way that the resulting game is substantially different from the symmetric ones, and does not produce the same equilibrium (although also in this case, the open-loop equilibrium is strongly time consistent).

Steady state equilibrium profits can be used to assess firms' incentives to adopt price or quantity as the relevant market variable, in the same vein as in Singh and Vives (1984). This highlights that, in the dynamic setting, quantity is no longer a (weakly) dominant strategy, as it is instead in the static setting. Indeed, there emerges that there exists a parameter region wherein the mixed price-quantity profile is an equilibrium one. Concerning social incentives, the main result is that the level of social welfare in the steady state associated with the asymmetric setting is larger than the steady state social welfare of the symmetric settings where both firms are price setters or both firms are quantity setters. The straightforward implication of this analysis is that the conflict between private and social incentives towards the adoption of prices or quantities may disappear in a dynamic model. This result modifies the conclusions about the social desirability concerning price or quantity setting, achieved by taking a static game perspective (see, e.g., Singh and Vives, 1984).

The remainder of the paper is structured as follows. Section 2 explains why the differential game approach is particularly appropriate to deal with the issue at hand, and briefly explains the basics of the different solution concepts for differential games. Section 3 develops the model, examining in turn (i) the features of demand and supply sides, (ii) the equilibrium under Cournot competition, (iii) the equilibrium under Bertrand competition, (iv) the equilibrium in the mixed setting. Section 4 compares the results for firms under the different settings, and considers the game where firms can choose whether to be price- or quantity-setters. Section 5 analyses social welfare across the steady state allocations originated by the different market competition regimes. Section 6 concludes.

2 The differential game approach

We believe that, in order to analyse the firms' behaviour concerning the accumulation of productive capacity, the differential game approach is particularly appropriate. Indeed, building capacity needs time; the depreciation of capacity over time is an important factor; the time dimension is important in planning the investment efforts towards capacity accumulation. Furhtermore, strategic interaction among firms is an important issue both in the market phase, and in the phase when capacity decisions are taken. Last but not least, the differential game approach leads to results encompassing those obtained by the more traditional approach based on static two-stage games. More precisely, the differential game approach allows to understand *under which particular conditions*, the results from static game can be interpreted as the steady state solution of the differential game.[1] Specifically, in the present paper we aim at checking whether the well-known result obtained by

[1] For further motivation supporting the choice of the differential game approach, see Dockner *et al.* (2000), chapter 9. This reference also provides a presentation of relevant differential games with capital accumulation.

Kreps and Scheinkman (1983) about the equivalence between the Cournot and the Bertrand setting in a two-stage static game, holds in a differential game framework too.

The existing literature on differential games mainly focusses on two kinds of strategies adopted by players: the open-loop and the closed-loop strategies.[2] When players adopt the open-loop solution concept, they design the time path concerning the control variable(s) at the initial time and then stick to it forever. This means that the open-loop strategy is simply a time path of actions, and time is the only determinant of the action to be done at any instant. The relevant equilibrium concept is the open-loop Nash equilibrium, which is only weakly time consistent and therefore, in general, it is not subgame perfect.[3] When players adopt the closed-loop strategy, they do not precommit control variable(s) on any path, and their actions at any instant may depend on the history of the game up to that instant, and, in particular, on the values of the state variables. In this situation, the information set used by players in setting their actions at any given time is often simplified to be only the current value of the state variables at that time, along with the initial conditions. This specific situation is labelled as *memoryless closed-loop* (Mehlmann, 1988). The relevant equilibrium concept, in this case, is the closed-loop memoryless Nash equilibrium, which is strongly time consistent (or subgame perfect).[4]

The literature on differential games devotes a considerable amount of attention to identifying classes of games where the closed-loop equilibria degenerate into open-loop equilibria. The degeneration means that the Nash equilibrium time paths of control variables coincide under the different strategy concepts. Whenever an open-loop equilibrium is a degenerate closed-loop equilibrium, then the former is also strongly time consistent (or subgame

[2] See Kamien and Schwartz (1981); Basar and Olsder (1982, 1995^2); Mehlmann (1988); Dockner et al. (2000). See also Cellini and Lambertini (2001), upon which the present Section is largely based.

[3] As to the definition of time consistency and subgame perfection, we rely - among many different definitions available in the literature - on the definition provided by Dockner *et al.* (2000, Section 4.3). We use "strong time consistency" as a synonomous of subgame perfection.

[4] Different rules of closed-loop class do exist: e.g., the perfect-memory closed loop, where the control variables depend on the complete history of the state variable(s); the feedback rule, where only the current stock of states are considered, irrespective of initial conditions. For oligopoly models where firms follow feedback rules, see Simaan and Takayama (1978), Fershtman and Kamien (1987, 1990), Dockner and Haug (1990), *inter alia*.

perfect). Therefore, one can rely upon the open-loop equilibrium which, in general, is much easier to derive than closed-loop.[5]

In what follows, we adopt the closed-loop memoryless solution concept, and we show that it collapses into the open-loop solution, in the sense that the Nash equilibrium time path of the variables under the two different concepts coincide. This is due to two features of the game we are going to present: first, the dynamic behaviour of any firm's state variable (i.e., capacity) does not depend on the rivals' control and state variables, which makes the kinematic equations concerning other firms redundant; second, for any firm, the first order conditions taken w.r.t. the control variables are independent of the rivals' state variables, which entails that the cross effect from rivals' states to own controls (which characterises the closed-loop information structure) disappears.

3 The model

3.1 The basic setup

The game is played over continuous time, $t \in [0, \infty)$.[6] In any instant of time, the market is served by two firms, 1 and 2, producing a differentiated good. Let $q_i(t)$ define the quantity sold by firm $i = 1, 2$ at time t. The marginal production cost is constant and equal to c for all firms. For the sake of simplicity, we pose $c = 0$. As in Singh and Vives (1984), the demand function for good i at time t is:

$$p_i(t) = A - q_i(t) - Dq_j(t) \ , (j \neq i) \tag{1}$$

where parameter $D \in [-1, 1]$ measures the degree of substitutability or complementarity between goods. If $D = 1$, they are perfect substitutes; if $D = 0$ the goods are independent and each firm behaves as a monopolist; if $D = -1$, products are perfect complements.

[5] Classes of games where this coincidence arises are illustrated in Clemhout and Wan (1974); Reinganum (1982); Mehlmann and Willing (1983); Dockner et al.(1985); Fershtman (1987); Fershtman et al. (1992). For an overview, see also Mehlmann (1988), and Dockner et al.(2000, ch. 7).

[6] The game can be reformulated in discrete time without significantly affecting its qualitative properties. For further details, see Basar and Olsder (1982, 1995^2).

In order to produce, firms must accumulate capacity or physical capital $k_i(t)$ over time. As in Solow (1956) and Nerlove and Arrow (1962), the relevant dynamic equation describing the accumulation of capacity (or physical capital) is:[7]

$$\frac{\partial k_i(t)}{\partial t} = I_i(t) - \delta k_i(t)\,, \tag{2}$$

where $I_i(t)$ is the investment carried out by firm i at time t, and $\delta > 0$ is the constant depreciation rate. Notice that the kinematic equation of player i's state variable is unaffected by the state and control variables of the rival. That is, strategic interaction among firms takes place through instantaneous profits only.[8]

The instantaneous cost of investment is $C_i\left[I_i\left(t\right)\right] = b\left[I_i\left(t\right)\right]^2$, with $b > 0$. We also assume that firms operate with a technology $q_i(t) = f(k_i(t))$. For the sake of analytical tractability, we assume that $f(k_i(t)) = k_i(t)$. Therefore, $q_i(t) = k_i(t)$. This equation may be interpreted by stating that each firm uses its whole capacity in order to produce output. This is quite obvious, provided that the capacity accumulation is costly. As a consequence, the demand function for variety i rewrites as:

$$p_i(t) = A - k_1(t) + Dk_2(t)\,. \tag{3}$$

Player i's objective is the maximisation of the present value of the profit flows:

$$\max_{I_i} \int_0^\infty \pi_i(.,t)e^{-\rho t}dt \tag{4}$$

subject to the dynamic constraint represented by the behaviour of the state variables (2) for $i = 1, 2$. Instantaneous profit is $\pi_i(.,t) = p_i q_i - b\left[I_i\left(t\right)\right]^2$. The factor $e^{-\rho t}$ discounts future gains, and the discount rate ρ is assumed to be constant and common to both players. In order to solve his optimisation problem, each player defines a strategy.

The control variable of firm i is the instantaneous investment $I_i(t)$, while $k_i(t)$ is obviously a state variable.

[7] For a dynamic Cournot game where capital accumulates *à la* Ramsey (i.e., current unsold output becomes additional productive capacity), see Cellini and Lambertini (1998).

[8] We follow this route in order to keep our models in line with the original formulations of dynamics (2). However, the analysis could be easily extended to account for the interaction between state and control variables of all players in the state dynamics without significantly changing our conclusions. A sufficient condition for all the ensuing results to continue to hold is that the kinematic equations of state variables be additively separable in state and control variables (see, e.g., Mehlmann, 1988, ch. 4).

3.2 Cournot competition

When firms write profits using the inverse demand functions (1), the closed-loop formulation of the Hamiltonian of firm i writes as follows:

$$\begin{aligned}\mathcal{H}_i(t) &= e^{-\rho t} \cdot \left\{ [A - k_i(t) - Dk_j(t)]\, k_i(t) - b\,[I_i\,(t)]^2 + \right. \\ &\left. +\lambda_{ii}(t)\,[I_i(t) - \delta k_i(t)] + \lambda_{ij}(t)\,[I_j(t) - \delta k_j(t)] \right\}\end{aligned} \tag{5}$$

where $\lambda_{ij}(t) = \mu_{ij}(t)e^{\rho t}$, and $\mu_{ij}(t)$ is the co-state variable associated to $k_j(t)$, $i, j = 1, 2$. Moreover, let $k_i(0) \equiv k_{i0}$ define the initial condition for firm i.

On the basis of (5), we can prove the following:

Proposition 1 *Under the Solow-Nerlove-Arrow capital accumulation dynamics, the open-loop Nash equilibrium is a degenerate closed-loop memoryless equilibrium. Therefore, the open-loop equilibrium is subgame perfect.*

Proof. Optimality conditions for firm $i = 1, 2$ require:

$$\begin{aligned}&(i)\ \frac{\partial \mathcal{H}_i(t)}{\partial I_i(t)} = 0 \Rightarrow -2bI_i(t) + \lambda_{ii}(t) = 0 \\ &(ii)\ -\frac{\partial \mathcal{H}_i(t)}{\partial k_i(t)} - \frac{\partial \mathcal{H}_i(t)}{\partial I_j(t)}\frac{\partial I_j^*(t)}{\partial k_i(t)} = \frac{\partial \lambda_{ii}(t)}{\partial t} - \rho\lambda_{ii}(t) \Rightarrow \\ &\Rightarrow -\frac{\partial \lambda_{ii}(t)}{\partial t} + \rho\lambda_{ii}(t) = A - 2k_i(t) - Dk_j(t) - \delta\lambda_{ii}(t) \\ &(iii)\ -\frac{\partial \mathcal{H}_i(t)}{\partial k_j(t)} - \frac{\partial \mathcal{H}_i(t)}{\partial I_j(t)}\frac{\partial I_j^*(t)}{\partial k_j(t)} = \frac{\partial \lambda_{ij}(t)}{\partial t} - \rho\lambda_{ij}(t) \\ &(iv) \lim_{t\to\infty} \mu_{ii}(t) \cdot k_i(t) = 0\ ;\ \lim_{t\to\infty} \mu_{ij}(t) \cdot k_j(t) = 0\ ,\end{aligned} \tag{6}$$

where (iv) are the transversality conditions.

The terms

$$\frac{\partial \mathcal{H}_i\,(.,t)}{\partial I_j(t)}\frac{\partial I_j^*(t)}{\partial k_i(t)} \tag{7}$$

appearing in the adjoint equations (6.ii,iii) capture the strategic interaction through the feedback from states to controls, which is by definition absent under the open-loop solution concept. $I_j^*(t)$ denotes the solution to the first order condition of firm j w.r.t. her control variable. Whenever the expression in (7) is zero, then the closed-loop memoryless equilibrium collapses into the open-loop Nash equilibrium, in the sense that the time path of all relevant variables under the two different solution rules coincide. This happens indeed in the present case, because by (6.i) we have: $\partial I_j^*(t)/\partial k_i(t) = 0$ for $i \neq j$,which means that the first order condition of a firm with respect to her control variable does not contain the state variables pertaining to the other different player. ■

Notice also that condition (6.iii), which yields $\partial \lambda_{ij}(t)/\partial t$, is redundant in that $\lambda_{ij}(t)$ does not appear in the first order conditions (6.i) and (6.ii). Therefore, we can simplify the problem, by setting $\lambda_{ij}(t) = 0$ for all $t \in [0, \infty)$ and $j \neq i$, and by setting $\lambda_{ii} = \lambda_i$ provided by only one co-state variable is relevant for any player.

Accordingly, the Hamiltonian simplifies as follows:[9]

$$\mathcal{H}_i(t) = e^{-\rho t} \cdot \left\{ \left[A - k_i(t) - Dk_j(t) \right] k_i(t) - b \left[I_i(t) \right]^2 + \lambda_i(t) \left[I_i(t) - \delta k_i(t) \right] \right\} . \tag{8}$$

Firm i's first order condition and the adjoint equations are (the transversality condition is omitted for brevity):

$$\frac{\partial \mathcal{H}_i(t)}{\partial I_i(t)} = 0 \Rightarrow \lambda_i(t) = 2bI_i(t) \tag{9}$$

$$\begin{aligned} -\frac{\partial \mathcal{H}_i(t)}{\partial k_i(t)} &= \frac{\partial \lambda_i(t)}{\partial t} - \rho \lambda_i(t) \Rightarrow \\ \frac{\partial \lambda_i(t)}{\partial t} &= (\rho + \delta) \lambda_i(t) - [A - 2k_i(t) - Dk_j(t)] \end{aligned} \tag{10}$$

Now we can explicitly look for steady state points. From the first order condition w.r.t. $I_i(t)$, we obtain:

$$\frac{\partial I_i(t)}{\partial t} = \frac{1}{2b} \frac{\partial \lambda_i(t)}{\partial t} = \frac{I_i(t)(\rho + \delta)}{2} - \frac{A - 2k_i(t) - Dk_j(t)}{2b} \tag{11}$$

Now, solving the system:

$$\frac{\partial I_i(t)}{\partial t} = 0 \, ; \; \frac{\partial k_i(t)}{\partial t} = 0 \, , \; i = 1, 2, \tag{12}$$

[9] The analysis of the open-loop equilibrium of a Solow-Cournot game with reversible investment similar to the one investigated here can be found in Fershtman and Muller (1984). For the feeback equilibrium, and its comparison with the open-loop equilibrium, see Reynolds (1987). See also Fudenberg and Tirole (1983).

we calculate the steady state levels of states and controls:

$$I^{ss} = \frac{\delta A}{2 + D + 2b(\rho + \delta)\delta} \; ; \; k^{ss} = \frac{I^{ss}}{\delta} = \frac{A}{2 + D + 2b(\rho + \delta)\delta} \, . \tag{13}$$

The pair $\{I^{ss}, k^{ss}\}$ is a saddle point; the proof is straightforward, in that, the determinant of the Jacobian matrix associated to the dynamic system $\{\partial k_i(t)/\partial t = 0, \partial I_i(t)/\partial t = 0\}$ is negative, while its trace is positive.[10]

Moreover, from (13) it can be shown that, in $\delta = 0$:

$$k^{ss} = q^{ss} = \frac{A}{2 + D} \tag{14}$$

which coincides with the equilibrium output of the static game studied by Singh and Vives (1984). In general, however:

Proposition 2 *For all positive and admissible values of parameters $\{\delta, \rho\}$, steady state capacity and sales are lower than in the static Cournot game with product differentiation.*

Steady state profits are:

$$\pi^{ss} = \frac{A^2 [1 + b\delta(2\rho + \delta)]}{\{2[1 + b\delta(\rho + \delta)] + D\}^2} \tag{15}$$

that, once again coincide with the Cournot-Nash equilibrium profits $\pi^{CN} = A^2/(2 + D)^2$ attained in the static model when $\delta = 0$. In general, the effects of parameters $\{\delta, \rho\}$ on steady state profits are described by the following partial derivatives:

$$\frac{\partial \pi^{ss}}{\partial \delta} = -\frac{2A^2 b \left[2b\delta^2(3\rho + \delta) + 2\delta(1 + b\rho^2) - D(\rho + \delta)\right]}{\{2[1 + b\delta(\rho + \delta)] + D\}^3} \; ; \tag{16}$$

$$\frac{\partial \pi^{ss}}{\partial \rho} = -\frac{2A^2 b\delta(2b\delta\rho - D)}{\{2[1 + b\delta(\rho + \delta)] + D\}^3} \, . \tag{17}$$

This simple comparative static exercise suffices to prove the following:

Proposition 3 *The steady state profits are non-monotone in both ρ and δ.*

[10] Calculations are trivial and they are omitted for brevity.

The above Proposition, in turn, implies that there exist parameter ranges wherein the steady state profits π^{ss} generated by the differential game are larger than the static equilibrium profits π^{CN}:

$$\pi^{ss} - \pi^{CN} \propto (2 + D) [\delta (D - 2) + 2D\rho] - 4b\delta (\rho + \delta)^2 , \tag{18}$$

with the r.h.s. of (18) being equal to zero at:

$$D = \frac{-2\rho \pm 2 (\rho + \delta) \sqrt{1 + b\delta (\delta + 2\rho)}}{\delta + 2\rho} . \tag{19}$$

The smaller root is negative, while the larger root is positive and smaller than one for all

$$b < \frac{3 (2\rho - \delta)}{4\delta (\delta + \rho)^2} \tag{20}$$

provided that $\rho > \delta/2$. This proves the following Corollary to Proposition 3:

Corollary 1 *Suppose $\rho > \delta/2$. If so, then $\pi^{ss} > \pi^{CN}$ for all*

$$D \in \left(\frac{-2\rho + 2 (\rho + \delta) \sqrt{1 + b\delta (\delta + 2\rho)}}{\delta + 2\rho} , 1 \right] .$$

3.3 Bertrand competition

With price competition *à la* Bertrand, from (1) the demand function firm i faces at time t is

$$q_i(t) = \frac{A}{1 + D} - \frac{p_i(t)}{1 - D^2} + \frac{Dp_j(t)}{1 - D^2} \tag{21}$$

where $p_i(t)$ and $p_j(t)$ are respectively the price set by firms i and j, respectively.

Under the capital accumulation rule (2), and using the assumption $q_i(t) = k_i(t)$, the closed-loop Hamiltonian of firm i is:

$$\begin{aligned} \mathcal{H}_i(t) = \; & e^{-\rho t} \left\{ \left[\frac{A}{1 + D} - \frac{p_i(t)}{1 - D^2} + \frac{Dp_j(t)}{1 - D^2} \right] p_i(t) - b [I_i (t)]^2 + \right. \\ & \lambda_{ii}(t) \left[I_i(t) - \delta \left(\frac{A}{1 + D} - \frac{p_i(t)}{1 - D^2} + \frac{Dp_j(t)}{1 - D^2} \right) \right] + \\ & \left. \lambda_{ij}(t) \left[I_j(t) - \delta \left(\frac{A}{1 + D} - \frac{p_j(t)}{1 - D^2} + \frac{Dp_i(t)}{1 - D^2} \right) \right] \right\} \end{aligned} \tag{22}$$

where $\lambda_{ij}(t) = \mu_{ij}(t)e^{\rho t}$, and $\mu_{ij}(t)$ is the co-state variable associated by firm i to the state variable $k_j(t)$.

The equivalent of Proposition 1 is easy to prove. That is, first order conditions on controls do not contain the state variables, and therefore the open-loop equilibrium is subgame perfect. The details are omitted for brevity.

Accordingly, we proceed by solving the open-loop formulation of the game, which obtains from (22) by setting $\lambda_{ij}(t) = 0$ and $\lambda_{ii}(t) = \lambda_i(t)$. The outcome is summarised by the following:

Proposition 4 *The steady state of the Solow-Nerlove-Arrow game with Bertrand competition is observationally equivalent to the steady state of the same game with Cournot competition.*

Proof. The first order condition on investment is:

$$\frac{\partial \mathcal{H}_i(t)}{\partial I_i(t)} = 0 \Rightarrow -2bI_i(t) + \lambda_i(t) = 0\,, \tag{23}$$

yielding

$$\lambda_i(t) = 2bI_i(t) \tag{24}$$

$$\frac{\partial I_i(t)}{\partial t} = \frac{1}{2b}\frac{\partial \lambda_i(t)}{\partial t} \tag{25}$$

In the Bertrand setting, deriving the co-state equation is more involved that in the Cournot setting. Since we are using direct demand functions, capacity $k_i(t)$ is expressed as a function of the price vector $\{p_i(t)\,,\, p_j(t)\}$. Therefore,

$$\frac{\partial \mathcal{H}_i(t)}{\partial k_i(t)} = \frac{\partial \mathcal{H}_i(t)}{\partial p_i(t)}\frac{\partial p_i(t)}{\partial k_i(t)} + \frac{\partial \mathcal{H}_i(t)}{\partial p_j(t)}\frac{\partial p_j(t)}{\partial k_i(t)} \tag{26}$$

where

$$\begin{aligned} &\frac{\partial \mathcal{H}_i(t)}{\partial p_i(t)} = \frac{A(1-D) - 2p_i(t) + Dp_j(t) + \delta\lambda_i(t)}{1-D^2}\,; \\ &\frac{\partial \mathcal{H}_i(t)}{\partial p_j(t)} = \frac{D\left[p_i(t) - \delta\lambda_i(t)\right]}{1-D^2}\,; \\ &\frac{\partial p_i(t)}{\partial k_i(t)} = -1\,;\; \frac{\partial p_j(t)}{\partial k_i(t)} = -D\,. \end{aligned} \tag{27}$$

Partial derivatives $\partial p_i/\partial k_i$ and $\partial p_i/\partial k_j$ are calculated using the inverse demand function (1). Using (27), the co-state equation writes as follows:

$$-\left[\frac{\partial \mathcal{H}_i(t)}{\partial p_i(t)}\frac{\partial p_i(t)}{\partial k_i(t)} + \frac{\partial \mathcal{H}_i(t)}{\partial p_j(t)}\frac{\partial p_j(t)}{\partial k_i(t)}\right] = \frac{\partial \lambda_i(t)}{\partial t} - \rho\lambda_i(t) \tag{28}$$

from which we obtain

$$\frac{\partial \lambda_i(t)}{\partial t} = \frac{A(1-D) - p_i(t)(2-D^2) + Dp_j(t) + \lambda_i(t)(\rho+\delta)(1-D^2)}{1-D^2} \,. \tag{29}$$

Then, plugging (24) and (29) into (25) and imposing the symmetry condition $p_j(t) = p_i(t)$, we have:

$$\frac{\partial I_i(t)}{\partial t} = \frac{A - p_i(t)(2+D) + 2bI_i(t)(\rho+\delta)(1+D)}{2b(1+D)} \tag{30}$$

with

$$\frac{\partial I_i(t)}{\partial t} = 0 \text{ at } I_i^{ss} = \frac{p_i(t)(2+D) - A}{2b(\rho+\delta)(1+D)} \,. \tag{31}$$

I_i^{ss} can be substituted into (2), which simplifies as follows:

$$\frac{\partial k_i(t)}{\partial t} = \frac{p_i(t)(2+D) - A}{2b(\rho+\delta)(1+D)} - \frac{\delta[A - p_i(t)]}{1+D} \tag{32}$$

with

$$\frac{\partial k_i(t)}{\partial t} = 0 \text{ at } p_i^{ss} = \frac{A[1 + 2b\delta(\rho+\delta)]}{2[1 + b\delta(\rho+\delta)] + D} \tag{33}$$

Now, using (33), we can simplify the expression for the steady state levels of investment and capacity:

$$I_i^{ss} = \frac{\delta A}{2 + D + 2b(\rho+\delta)\delta} \;;\; k_i^{ss} = \frac{I_i^{ss}}{\delta} \,, \tag{34}$$

which coincide with (13). Also the equilibrium profits are obviously the same as in the Cournot game investigated in section 3.1. ■

The above result has the following intuitive explanation. The usual interpretation of the difference between Cournot and Bertrand in static games is that firms optimise w.r.t. either quantities or prices. However, in a differential game, using direct demand functions for the Bertrand case does not modify the strategy space for control variables, which are investment efforts. Given that accumulation efforts entail a positive cost, capacity has to be fully used under equilibrium conditions. Therefore, in the present theoretical framework, firms *are not* choosing prices or quantities and consequently the specific formulation of instantaneous profits is immaterial to the equilibrium emerging in steady state. Nevertheless, this conclusion was not obvious at the outset, in that inverting demand functions involves a reformulation of the dynamics of state variables as well as the co-state equations. The observational equivalence between Bertrand and Cournot outcomes is due to the following reasons: (i) the strategy space for control variables is the same in the two cases; and (ii) the state and co-state equations of the Cournot model transform into the corresponding state and co-state equations of the Bertrand model through a symmetric affine transformation. This entails that the dynamic equations of controls are also the same in the two settings.

Proposition 6 has a relevant corollary:

Corollary 2 *In the Bertrand formulation of the Nerlove-Arrow-Solow game, the steady state price is the same as in the Cournot formulation of the game.*

That is, when capital (or capacity) accumulates according to (2), and the whole capacity is used to produce output, the Bertrand paradox never arises. This is easily shown by verifying that p_i^{ss} in (33) is always strictly higher than marginal cost c for all admissible values of parameters. In particular, if $D = 1$, then

$$p_i^{ss} = \frac{A\left[1 + 2b\delta\left(\rho + \delta\right)\right]}{3 + 2b\delta\left(\rho + \delta\right)} \tag{35}$$

which becomes

$$p_i^{ss} = \frac{A}{3} \tag{36}$$

when $\delta = 0$. That is, when products are perfect substitutes and capital does not depreciate, the steady state price coincides with the well known equilibrium price associated with the static version of Cournot duopoly.[11] The same obviously holds for capacity (and output), $k_i^{ss} = A/3$.

This means that the Solow-Nerlove-Arrow model generalises the static two-stage game *à la* Kreps and Scheinkman (1983), where firms first choose capacities and then compete in prices. Indeed, the present model encompasses Kreps and Scheinkman's, with no need of resorting to mixed strategies, as it produces the Cournot equilibrium as the subgame perfect capacity-constrained equilibrium of a differential game in prices and investments in pure strategies.

[11] It is worth noting that this holds for all admissible values of the discount rate ρ, provided that $D = 1$ and $\delta = 0$. On the contrary, it is easily shown that the static Cournot outcome does not hold in the limit cases where $\rho = 0$ or $\rho \to \infty$.

4 The mixed setting

Consider now the situation where firm 1 is a quantity setter while firm 2 is a price setter. The relevant demand functions are respectively:

$$p_1(t) = A(1-D) - k_1(t)\left(1-D^2\right) + Dp_2(t)\ ; \tag{37}$$

$$k_2(t) = A - Dk_1(t) - p_2(t)\ . \tag{38}$$

The state equations write as follows:

$$\frac{\partial k_1(t)}{\partial t} = I_1(t) - \delta k_1(t)\,; \tag{39}$$

$$\frac{\partial k_2(t)}{\partial t} = I_2(t) - \delta\left[A - Dk_1(t) - p_2(t)\right]\ . \tag{40}$$

The closed-loop Hamiltonians are:

$$\begin{aligned}\mathcal{H}_1(t) = \quad & e^{-\rho t}\left\{\left[A(1-D) - k_1(t)\left(1-D^2\right) + Dp_2(t)\right]q_1(t) - b\left[I_1(t)\right]^2 + \right.\\ & \left.\lambda_{11}(t)\left[I_i(t) - \delta k_1(t)\right] + \lambda_{12}(t)\left[I_2(t) - \delta\left(A - Dk_1(t) - p_2(t)\right)\right]\right\}\end{aligned} \tag{41}$$

$$\begin{aligned}\mathcal{H}_2(t) = \quad & e^{-\rho t}\left\{\left[A - Dk_1(t) - p_2(t)\right]p_2(t) - b\left[I_2(t)\right]^2 + \right.\\ & \left.\lambda_{22}(t)\left[I_2(t) - \delta\left(A - Dk_1(t) - p_2(t)\right)\right] + \lambda_{21}(t)\left[I_1(t) - \delta k_1(t)\right]\right\}\end{aligned} \tag{42}$$

The equivalent of Proposition 1 can be shown to hold here as well.. The details are omitted for the sake of brevity. Moreover, one can solve the open-loop Hamiltonians setting (i) $\lambda_{ij}(t) = 0$ for all $t \in [0,\infty)$ and $j \neq i$; and (ii) $\lambda_{ii}(t) = \lambda_i(t)$. The first order conditions on controls are:

$$\frac{\partial \mathcal{H}_i(t)}{\partial I_i(t)} = \lambda_i(t) - 2bI_i(t) = 0\ , \tag{43}$$

entailing:

$$\lambda_i(t) = 2bI_i(t)\ ;\ \frac{\partial I_i(t)}{\partial t} \propto \frac{\partial \lambda_i(t)}{\partial t}\ . \tag{44}$$

As to the co-state equations, one has to proceed in the following way. While the co-state equation of the quantity setter (firm 1) is standard:

$$-\frac{\partial \mathcal{H}_1(t)}{\partial k_1(t)} = \frac{\partial \lambda_1(t)}{\partial t} - \rho\lambda_1(t) \Rightarrow \tag{45}$$

$$\frac{\partial \lambda_1(t)}{\partial t} = (\rho + \delta)\,\lambda_1(t) - \left[A\,(1-D) - 2\left(1-D^2\right)k_1(t) + Dp_2(t)\right]\ , \qquad (46)$$

the derivation of the co-state equation for the price setter (firm 2) is slightly more involved, since in the demand function (38) capacity $k_2(t)$ has to be expressed as a function of the vector $\{k_1(t)\,,\,p_2(t)\}$. Therefore:

$$\frac{\partial \mathcal{H}_2(t)}{\partial k_2(t)} = \frac{\partial \mathcal{H}_2(t)}{\partial p_2(t)}\frac{\partial p_2(t)}{\partial k_2(t)} + \frac{\partial \mathcal{H}_2(t)}{\partial k_1(t)}\frac{\partial k_1(t)}{\partial k_2(t)} \qquad (47)$$

where

$$\begin{aligned}
&\frac{\partial \mathcal{H}_2(t)}{\partial p_2(t)} = A + \delta\lambda_2 - Dk_1(t) - 2p_2(t)\ ;\\
&\frac{\partial \mathcal{H}_2(t)}{\partial k_1(t)} = D\left[\delta\lambda_2(t) - p_2(t)\right]\ ;\\
&\frac{\partial p_2(t)}{\partial k_2(t)} = -1\ ;\ \frac{\partial k_1(t)}{\partial k_2(t)} = 0\ .
\end{aligned} \qquad (48)$$

Partial derivatives $\partial p_2(t)/\partial k_2(t)$ and $\partial k_1(t)/\partial k_2(t)$ are calculated using (38) and (37), respectively; $\partial k_1(t)/\partial k_2(t) = 0$ since (37) reveals that $k_1(t)$ does not depend upon $k_2(t)$. Using (48), the co-state equation of firm 2 writes as follows:

$$-\frac{\partial \mathcal{H}_2(t)}{\partial p_2(t)}\frac{\partial p_2(t)}{\partial k_2(t)} = \frac{\partial \lambda_2(t)}{\partial t} - \rho\lambda_2(t) \Rightarrow \qquad (49)$$

$$\frac{\partial \lambda_2(t)}{\partial t} = (\rho + \delta)\,\lambda_2(t) + A - Dk_1(t) - 2p_2(t)\,. \qquad (50)$$

Now it is worth stressing that (46) and (50), the system of differential equations $\{\partial I_i(t)/\partial t = 0\}$ is asymmetric. Although the strategy space for controls is by definition the same as in the symmetric games investigated above, the dynamic behaviour of controls is now asymmetric across firms. Consequently, we cannot expect to observe symmetric investment and capacity levels in steady state. This suffices to prove that the mixed setting cannot reproduce the Cournot outcome.

On the basis of (46), (50) and (44), we have:

$$\frac{\partial I_1(t)}{\partial t} \propto 2b\,(\rho + \delta)\,I_1(t) - A(1-D) + 2\left(1-D^2\right)k_1(t) - Dp_2(t) \qquad (51)$$

and

$$\frac{\partial I_2(t)}{\partial t} \propto 2b\,(\rho + \delta)\,I_2(t) + A - 2p_2(t) - Dk_1(t)\,. \qquad (52)$$

Solving the system $\{\partial I_i(t)/\partial t = 0\}$, we obtain the following expression (the indication of time is dropped henceforth):

$$\begin{aligned} I_1^{ss} &= \frac{A(1-D) - 2\left(1-D^2\right)k_1 + Dp_2}{2b\left(\rho+\delta\right)}\ ; \\ I_2^{ss} &= -\frac{A - Dk_1 - 2p_2}{2b\left(\rho+\delta\right)}\ , \end{aligned} \tag{53}$$

that can be used to simplify the state equations (39-40). Then, the linear system $\{\partial k_i(t)/\partial t = 0\}$ can be solved w.r.t. k_1 and p_2 :

$$k_1^{ss} = \frac{A\left\{2\left[1+b\delta\left(\rho+\delta\right)\right] - D\right\}}{4\left[1+b\delta\left(\rho+\delta\right)\right]^2 - D^2\left[3+2b\delta\left(\rho+\delta\right)\right]} \tag{54}$$

$$p_2^{ss} = \frac{A\left[1+2b\delta\left(\rho+\delta\right)\right]\left[2b\delta\left(\rho+\delta\right) + \left(1-D\right)\left(2+D\right)\right]}{4-3D^2+2b\delta\left(\rho+\delta\right)\left\{2\left[2+b\delta\left(\rho+\delta\right)\right] - D^2\right\}} \tag{55}$$

which, in turn, imply that the price setter holds the following capacity in steady state:

$$k_2^{ss} = \frac{A\left[2b\delta\left(\rho+\delta\right) + \left(1-D\right)\left(2+D\right)\right]}{4\left[1+b\delta\left(\rho+\delta\right)\right]^2 - D^2\left[3+2b\delta\left(\rho+\delta\right)\right]}\ , \tag{56}$$

while firm 1's equilibrium price is:

$$p_1^{ss} = \frac{A\left\{2\left[1+b\delta\left(\rho+\delta\right)\right] - D\right\}\left[1+2b\delta\left(\rho+\delta\right) - D^2\right]}{4\left[1+b\delta\left(\rho+\delta\right)\right]^2 - D^2\left[3+2b\delta\left(\rho+\delta\right)\right]}\ . \tag{57}$$

Equilibrium investments in steady state can be quickly written as $I_i^{ss} = \delta k_i^{ss}$. It can be easily checked that the above list of equilibrium magnitudes is admissible (i.e., non-negative) for all admissible values of parameter D. Again, one can easily verify that steady state points are stable in the saddle sense.

In the absence of capital depreciation, i.e., when $\delta = 0$, steady state capacities coincide with the equilibrium outputs of the static game (Singh and Vives, 1984):

$$k_1^{ss}|_{\delta=0} = \frac{A\left(2-D\right)}{4-3D^2}\ ;\ k_2^{ss}|_{\delta=0} = \frac{A\left(2-D-D^2\right)}{4-3D^2}\ , \tag{58}$$

with $k_1^{ss} > k_2^{ss}$ for all $D \in (0,1]$, and conversely for all $D \in [-1,0)$.

Steady state profits are respectively:

$$\pi_1^{ss}|_{\delta=0} = \frac{A^2 \{2[1+b\delta(\rho+\delta)] - D\}^2 [1+b\delta(2\rho+\delta) - D^2]}{\{4-3D^2+2b\delta(\rho+\delta)[2(2+b\delta(\rho+\delta)) - D^2]\}^2} \tag{59}$$

$$\pi_2^{ss}|_{\delta=0} = \frac{A^2 \{2[1+b\delta(\rho+\delta)] - D(1+D)\}^2 [1+b\delta(2\rho+\delta)]}{\{4-3D^2+2b\delta(\rho+\delta)[2(2+b\delta(\rho+\delta)) - D^2]\}^2} \tag{60}$$

which of course coincide with the static Nash equilibrium profits at $\delta = 0$:

$$\pi_1^N = \frac{A^2 (2-D)^2 (1-D^2)}{(4-3D^2)} \; ; \; \pi_2^N = \frac{A^2 (2-D-D^2)^2}{(4-3D^2)} \, . \tag{61}$$

5 Price or quantity?

Now we are in a position to examine the issue of choosing between price and quantity. In the present framework, where firms do not optimise profits w.r.t. market variables but only w.r.t. investment efforts, the choice between price and quantity refers instead to alternative ways of specifying the state variables as well as the state and co-state equations.

This issue can be investigated on the basis of Matrix 1, where the payoffs are given by steady state profits. In particular, define as π^{sym} the steady state equilibrium profits associated with either the Cournot or the Bertrand game (*sym* therefore stands for *symmetric*); as π^{kp} the profits of the quantity setter and π^{pk} the profits of the price setter in the asymmetric settings.

		firm 2: p	firm 2: k
firm 1	p	π^{sym} ; π^{sym}	π^{pk} ; π^{kp}
	k	π^{kp} ; π^{pk}	π^{sym} ; π^{sym}

Matrix 1

$$\begin{aligned} \pi^{sym} &= \frac{A^2 [1+b\delta(2\rho+\delta)]}{\{2[1+b\delta(\rho+\delta)] + D\}^2} \\ \pi^{kp} &= \frac{A^2 \{2[1+b\delta(\rho+\delta)] - D\}^2 [1+b\delta(2\rho+\delta) - D^2]}{\{4-3D^2+2b\delta(\rho+\delta)[2(2+b\delta(\rho+\delta)) - D^2]\}^2} \\ \pi^{pk} &= \frac{A^2 \{2[1+b\delta(\rho+\delta)] - D(1+D)\}^2 [1+b\delta(2\rho+\delta)]}{\{4-3D^2+2b\delta(\rho+\delta)[2(2+b\delta(\rho+\delta)) - D^2]\}^2} \end{aligned} \tag{62}$$

The case where $\delta = 0$ (irrespective of whether ρ is nil or positive) can be quickly dealt with, on the basis of the following inequalities:

$$\begin{aligned} \pi^{sym} &> \pi^{pk} \text{ for all } D \in [-1, 1] \\ \pi^{sym} &> \pi^{kp} \text{ for all } D \in (0, 1] \\ \pi^{sym} &> \pi^{pk} \text{ for all } D \in [-1, 0) \end{aligned} \tag{63}$$

For any given $\delta > 0$, a trivial case is that where ρ tends to infinity. If so, all equilibrium profits tend to zero and the choice between k and p becomes meaningless. If, on the contrary, $\rho = 0$, then for any admissible $\delta \in (0, 1]$, the nature of the game and the resulting equilibria depend upon parameter D, as it is quickly established by the following inequalities:

$$\pi^{sym} - \pi^{kp}\big|_{\rho=0} = \frac{A^2 D^4 \left[4b\left(\delta\left(1 + b\delta^2\right)\right) + D^2\right]}{\left[2\left(1 + b\delta^2\right) + D\right]^2 \left[4\left(1 + b\delta^2\right)^2 - D^2\left(3 + 2b\delta^2\right)\right]^2} > 0 \tag{64}$$

always, while

$$\pi^{pk} - \pi^{sym}\big|_{\rho=0} = -\frac{A^2\left(1 + b\delta^2\right) D^3 \left[8\left(1 + b\delta^2\right)^2 - D^2\left(6 + 4b\delta^2 + D\right)\right]}{\left[2\left(1 + b\delta^2\right) + D\right]^2 \left[4\left(1 + b\delta^2\right)^2 - D^2\left(3 + 2b\delta^2\right)\right]^2} \tag{65}$$

is negative for all $D \in (0, 1]$, and conversely for all $D \in [-1, 0)$.[12] On this basis, we can write:

Remark 1 *Consider the cases where either $\delta = 0$, or $\delta \in (0, 1]$ and $\rho = 0$. In both cases:*

- *In the range of substitutes, i.e., for all $D \in (0, 1]$, there exists no dominant strategy for either firm, and the (coordination) game has two symmetric equilibria in pure strategies: $\{p, p\}$ and $\{k, k\}$. Moreover,*

[12] Note that $8\left(1 + b\delta^2\right)^2 - D^2\left(6 + 4b\delta^2 + D\right) = 0$ at

$$b = \frac{D^2 - 4 \pm D\sqrt{4 + 2D + D^2}}{4\delta^2}$$

with both roots being always negative in the admissible range of parameter D. Therefore, $sign\left\{\pi^{pk} - \pi^{sym}\big|_{\rho=0}\right\} = sign\{-D\}$.

it also produces a correlated equilibrium, and a mixed strategy equilibrium where the probability of observing an asymmetric outcome $\{k, p\}$ *is strictly positive.*

- *In the range of complements, i.e., for all* $D \in [-1, 0)$, p *is a strictly dominant strategy for both firms. Therefore,* $\{p, p\}$ *is the unique equilibrium in pure strategies.*

Consider now the case where δ and ρ take positive and finite values. By resorting to numerical calculations, it can be verified that:

$$\begin{aligned} \pi^{sym} &> \pi^{pk} \text{ for all } D \in (0,1] \\ \pi^{sym} &< \pi^{pk} \text{ for all } D \in [-1,0) \end{aligned} \tag{66}$$

Moreover:

$$\pi^{sym} < \pi^{kp} \text{ for all } D \in (D_1, D_2) \tag{67}$$

while the opposite holds for all D outside the interval $[D_1, D_2]$. The values of D_1 and D_2 can be computed numerically for any admissible pair $\{\delta, \rho\}$, to ascertain that $D_1 = -D_2$, with $-1 < D_1 < 0 < D_2 < 1$. For instance, set $a = 1$, $b = 1$, and $\delta = 1/10$; then, for $\rho = 2\delta$ one obtains $D_1 = -D_2 \simeq -0.6798$, while for $\rho = \delta/4$ one obtains $D_1 = -D_2 \simeq -0.4136$. Note that, as ρ decreases, the interval $[D_1, D_2]$ shrinks.

This discussion produces the following:

Remark 2 *Given* $\delta \in (0, 1]$ *and* $\rho > 0$, *then:*

- *For all* $D \in (D_2, 1]$, *we observe a coordination game with two pure-strategy symmetric equilibria,* $\{k, k\}$ *and* $\{p, p\}$. *There also exist a correlated equilibrium and a mixed strategy equilibrium.*

- *For all* $D \in (0, D_2)$, k *is a dominant strategy for both firms. Therefore,* $\{k, k\}$ *is the unique pure-strategy equilibrium.*

- *For all* $D \in (D_1, 0)$, *we observe a chicken game with two pure strategy equilibria,* $\{k, p\}$ *and* $\{p, k\}$. *There also exist a correlated equilibrium and a mixed strategy equilibrium.*

- *For all* $D \in [-1, D_2]$, p *is a dominant strategy for both firms. Therefore,* $\{p, p\}$ *is the unique pure-strategy equilibrium.*

In the next Section, we carry out the welfare analysis at the level of the single market under consideration, and compare the social desirability of the different settings with the choices of the firms.

6 Welfare analysis

Singh and Vives (1984) show that the market demand function (1), is consistent with the behaviour of a representative consumer who maximises the function $S = U - (p_1q_1 + p_2q_2)$ with $U = [A(q_1 + q_2) - ({q_1}^2 + {q_2}^2 + Dq_1q_2)/2]$. Intuitively, S is a measure for the net consumer surplus, while the quadratic function U is the non-linear part of a quasi-linear utility function, where a numeraire good enters linearly. This formulation of consumer preferences makes the demand for goods q_1, q_2 inelastic to income, provided that the total income is larger than $p_1q_1 + p_2q_2$. Expression S is an obvious candidate to measure the consumer welfare in the present partial analysis framework. The instantaneous social welfare index, at the single market level, can be consistently defined as $SW = S + \pi_1 + \pi_2$.We confine our attention to the steady state allocations. Remembering that $q_i = k_i$, and $I_i = \delta k_i$ in steady state, social welfare in steady state can be written as follows:

$$SW = A(k_1 + k_2) - (1 + 2b\delta^2)(k_1^2/2 + k_2^2/2) - Dk_1k_2/2 \qquad (68)$$

where k_i has to be interptreted as the capacity of firm i in steady state. Simple substitutions of expressions given by (13) or (34), (54), and (56) into (68) allow to compare the welfare indexes across the steady states produced by different types of market competition. We are particularly interested in evaluating whether the welfare in steady state is larger in a symmetric setting (i.e., when either both firms compete either *à la* Cournot or both firms compete *à la* Bertrand) or in the mixed setting, that is, in the case where one firm behaves as a quantity-setter and the other behaves as a price-setter.

It is worth remembering that, in the static game, Cournot (Bertrand) competition is better for firms, if goods are substitutes (complements), while Bertrand competition is the best setting for a social planner maximising the social welfare, regardless of the nature the goods (substitutes or complements), as established by Singh and Vives (1984, p. 553). In the present model, the symmetric settings are observationally equivalent, so that the social desirability associated to these settings is exactly the same. Conversely, the social welfare is different in the asymmetric framework.

We denote the social welfare in the steady state originated by the Nash equilibrium under a symmetric setting as SW^{sym}, and the social welfare in the steady state originated by the Nash equilibrium under the asymmetric setting as SW^{kp}. We are interested in comparing the levels of social welfare. To this end, we evaluate the sign of $DSW = SW^{sym} - SW^{kp}$. Index DSW clearly depends on 5 parameters: A, b, D, ρ, δ. A complete analytical study of function DSW seems to be out of reach, with the exception of some particular cases. Specifically, when $\rho = 0$,it is:

$$DSW|_{\rho=0} = -2A^2D^2(D^3 - 2D^2 - D + 2)/[(D+2)(3D^2 - 4)^2] \qquad (69)$$

which is clearly always negative, except for $D = 0$ and $D = \pm 1$, where $DSW = 0$. This leads us to conclude that the setting providing the larger social welfare in steady state is the mixed setting, where one firm plays *à la* Cournot and the other one plays *à la* Bertrand. This conclusion holds when goods are both substitutes and complements. When goods are independent, firms behave as monopolists, and the choice concerning price or quantity is pointless, so that the social welfare associated to the different settings is the same. Formally, $D = 0$ always implies $DSW = 0$.

Taking into account the conclusions about the equilibrium choices of firms between price and quantity, we can resume the different cases that can occur at equilibrium:

Remark 3 *The welfare assessment of steady state equilibria can be summarised as follows:*

- *If, alternatively, (a)* $\delta \in (0,1]$, $\rho > 0$ *and* $D \in (D_2, 1]$, *(b) either* $\delta = 0$ *or* $\rho = 0$, *and* $D \in (0,1]$: *the (coordination) game has two symmetric Nash equilibria in pure strategies:* $\{p,p\}$ *and* $\{k,k\}$; *none of them is efficient from the social welfare perspective. Nevertheless, the socially efficient outcome* $\{k,p\}$ *arises with a strictly positive probability, even if it is not a pure-strategy Nash equilibrium in the game where firms choose between price and quantity setting.*

- *If, alternatively, (a)* $\delta \in (0,1]$, $\rho > 0$, *and* $D \in [-1, D_2]$, *(b) either* $\delta = 0$ *or* $\rho = 0$, *and* $D \in [-1,0)$: *there exists one Nash equilibrium in pure strategies,* $\{p,p\}$,*which is socially inefficient.*

- *If* $\delta \in (0,1]$, $\rho > 0$, *and* $D \in (0, D_2)$: *there exists one Nash equilibrium in pure strategies,*$\{k, k\}$*,which is socially inefficient.*

- *If* $\delta \in (0,1]$, $\rho > 0$, *and* $D \in (D_1, 0)$: *the (chicken) game has two Nash equilibria in pure strategies,*$\{k, p\}$ *and* $\{p, k\}$ *;both equilibria are efficient from the social standpoint. However, the probability of the observing a socially inefficient outcome is strictly positive.*

It is worth stressing, once again, that the welfare considerations refer to the steady state allocations. As one can see, the possibilities are much more articulated than in the static framework considered by Singh and Vives (1984). In particular, it is no longer true that the Bertrand setting is the best one from the social welfare point of view. Thus, we have to conclude that, in this case, the static game representation may not be a good simplified version (or reduced form) of a dynamic setting.

7 Conclusions

We have taken a differential game approach to study capacity accumulation and market behaviour in a duopoly with differentiated goods. Following a well established strand in the existing literature, the dynamic rule concerning capacity accumulation has been designed in such a way that any firm's state variable (i.e., the capacity) does not depend directly on the rivals' control and state variables. This is part of the motivation for the fact that the closed-loop memoryless Nash equilibria collapse into the open-loop Nash equilibria; as a consequence, the former is - in the present model - strongly time consistent.

We have solved the differential game, taking into account different descriptions of firms' market behaviour. Cournot competition, where both firms take quantity as the relevant market variable; Bertrand competition, where both firms take price as the relevant market variable; and the mixed setting, where one firm considers the quantity as the relevant market variable, while the other firm considers the price as the relevant variable.

We have shown that the Cournot and the Bertrand settings produce the same Nash equilibrium, as long as the strategy space for control variables is the same in the two cases (specifically, the space of capacity accumulation efforts), and the relevant market variable is immaterial to the optimal choice. This result confirms the well-known findings of Kreps and Scheinkman (1983) in a static two-stage game framework. Conversely, in the asymmetric setting, the Nash equilibrium produces a different outcome as compared to the symmetric settings.

Nash equilibrium solutions give rise to economically meaningful steady state points. Such steady states are stable in the saddle sense. We have carried out comparative evaluations across the steady state allocations associated with the different settings, and in particular we have examined the profits of firms and the social welfare indexes in steady state. If we allow firms to choose between price- and quantity-setting behaviour, the relevant payoffs being represented by steady state profits, several equilibrium profiles may emerge, depending on parameter configurations. The Nash equilibrium can be socially efficient or not; the socially efficient situation may be a Nash equilibrium or not; the socially efficient equilibrium may arise in association with mixed-strategy equilibria. Thus, our conclusions are markedly different from the very clear-cut results obtained in a static two-stage game framework (see, e.g., Singh and Vives, 1984), where firms always prefer Cournot (respectively, Bertrand) competition if they produce substitutes (resp., complements), while Bertrand competition is always preferable to Cournot competition in terms of social welfare. The substantial difference among the conclusions deriving from the static and the dynamic game approaches in this case, clearly shows that the static game is far from being a reduced-form description of dynamic competition among firms.

References

[1] Basar, T., and Olsder, G.J. (1982, 1995^2), *Dynamic Noncooperative Game Theory*, San Diego, Academic Press.

[2] Cellini, R. and L. Lambertini (1998), "A Dynamic Model of Differentiated Oligopoly with Capital Accumulation", *Journal of Economic Theory*, **83**, 145-55.

[3] Cellini, R. and L. Lambertini (2001), "Differential oligopoly Games where the Closed-Loop Memoryless and Open-Loop Equilibria Coincide", working paper, Department of Economics University of Bologna, n. 402.

[4] Clemhout, S. and H.Y. Wan, Jr. (1974), "A Class of Trilinear Differential Games", *Journal of Optimization Theory and Applications*, **14**, 419-24.

[5] Dockner, E.J., G. Feichtinger and S. Jørgensen (1985), "Tractable Classes of Nonzero-Sum Open-Loop Nash Differential Games: Theory and Examples", *Journal of Optimization Theory and Applications*, **45**, 179-97.

[6] Dockner, E.J. and A.A. Haug (1990), "Tariffs and Quotas under Dynamic Duopolistic Competition", *Journal of International Economics*, **29**, 147-59.

[7] Dockner, E.J, S. Jørgensen, N. Van Long and G. Sorger (2000), *Differential Games in Economics and Management Science*, Cambridge, Cambridge University Press.

[8] Edgeworth, F. (1897), "La Teoria Pura del Monopolio", *Giornale degli Economisti*, **40**, 13-31. Reprinted in English as "The Pure Theory of Monopoly", in F. Edgeworth (1925), *Papers Relating to Political Economy*, Vol. 1, London, Macmillan & Co., Ltd., 111-42.

[9] Fershtman, C. (1987), "Identification of Classes of Differential Games for Which the Open-Loop is a Degenerated Feedback Nash Equilibrium", *Journal of Optimization Theory and Applications*, **55**, 217-31.

[10] Fershtman, C. and M.I. Kamien (1987), "Dynamic Duopolistic Competition with Sticky Prices", *Econometrica*, **55**, 1151-64.

[11] Fershtman, C. and M.I. Kamien (1990), "Turnpike Properties in a Finite-Horizon Differential Game: Dynamic Duopoly with Sticky Prices", *International Economic Review*, **31**, 49-60.

[12] Fershtman, C. and E. Muller (1984), "Capital Accumulation Games of Infinite Duration", *Journal of Economic Theory*, **33**, 322-39.

[13] Fershtman, C., M. Kamien and E. Muller (1992), "Integral Games: Theory and Applications", in Feichtinger, G. (ed.), *Dynamic Economic Models and Optimal Control*, Amsterdam, North-Holland, 297-311.

[14] Fudenberg, D. and J. Tirole (1983), "Capital as a Commitment: Strategic Investment to Deter Mobility", *Journal of Economic Theory*, **31**, 227-50.

[15] Kamien, M.I. and N.L. Schwartz (1981), *Dynamic optimization: The Calculus of Variations and Optimal Control in Economics and Management*, Amsterdam, North-Holland.

[16] Kreps, D. and J. Scheinkman (1983), "Quantity Precommitment and Bertrand Competition Yield Cournot Outcomes", *Bell Journal of Economics*, **14**, 326-37.

[17] Levitan, R. and M. Shubik (1972), "Price Duopoly and Capacity Constraints", *International Economic Review*, **13**, 111-23.

[18] Mehlmann, A. (1988), *Applied Differential Games*, New York, Plenum Press.

[19] Mehlmann, A. and R. Willing (1983), "On Nonunique Closed-Loop Nash Equilibria for a Class of Differential Games with a Unique and Degenerate Feedback Solution", *Journal of Optimization Theory and Applications*, **41**, 463-72.

[20] Nerlove, M. and K.J. Arrow (1962), "Optimal Advertising Policy under Dynamic Conditions", *Economica*, **29**, 129-42.

[21] Osborne, M. and C. Pitchik (1986), "Price Competition in a Capacity-Constrained Duopoly", *Journal of Economic Theory*, **38**, 238-60.

[22] Reinganum, J. (1982), “A Class of Differential Games for Which the Closed Loop and Open Loop Nash Equilibria Coincide”, *Journal of Optimization Theory and Applications*, **36**, 253-62.

[23] Reynolds, S.S. (1987), “Capacity Investment, Preemption and Commitment in an Infinite Horizon Model”, *International Economic Review*, **28**, 69-88.

[24] Simaan, M. and T. Takayama (1978), “Game Theory Applied to Dynamic Duopoly Problems with Production Constraints”, *Automatica*, **14**, 161-66.

[25] Singh, N. and X. Vives (1984), “price and Quantity Competition in a Differentated Duopoly”, *Rand Journal of Economics*, **15**, 546-54.

[26] Solow, R. (1956),“A Contribution to the Theory of Economic Growth”, *Quarterly Journal of Economics*, **70**, 65-94.