A Simple Approach to CAPM and Option Pricing

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Abstract.
In this paper we propose a simple approach to asset valuation in terms of two characteristics, expected value and expected variability, and their distinct marginal contributions to the value of the market portfolio. The result is shown to correspond to Sharpe's CAPM. We then show that pricing in terms of characteristics (or CAPM) applies to any asset and in particular to option valuation. A pricing formula corresponding to Black and Scholes' no-arbitrage option pricing is obtained under the assumption of normal asset price distributions.

0. Introduction

Capital asset pricing model and option pricing theory: two of the best known and most important results of finance concern the *pricing* of assets. The first model is attributed to William Sharpe (1964) even if Tobin (1958), Treynor (1965), Lintner (1965) and Mossin (1966) reached similar results in the same years and all of them are in debt of Markowitz (1952, 1959) portfolio model.

Option pricing theory, instead, stems from the seminal paper of Black and Scholes (1973), in which an arbitrage argument is developed to solve in a new manner the old problem of pricing option contracts[1].

In both cases pricing is the relevant point at issue and two questions cannot be avoided. 1) In microeconomic theory, prices are marginal values (marginal cost and marginal utility, in equilibrium). Is the same marginal approach still valid in finance? 2) Notwithstanding the apparent differences, is there a unique pricing function containing both models?

As we shall see, the answer is yes to both questions.

In fact, it can be shown that the pricing function of an asset can be obtained, in a two parameter, normal approach, from the marginal contributions provided by the asset in terms of risk and return. Moreover, even if the two results appear quite different, a two parameter, normal approach is common to both and the two models can be obtained jointly: CAPM is able to price options and option prices, in a normal world[2], satisfies CAPM conditions.

1. The general mean-variance framework

Let us assume that investors are interested only in risk and return, considered as the two essential characteristics or factors of single assets and portfolios. Quadratic utility or normal distribution are the alternative hypotheses used to justify the mean-variance approach.

More precisely, in our view, the first assumption is that asset prices are determined by two factors (one positive and one negative). Market prices reflect price and quantity of each factor: price times quantity summed over all characteristics gives the market price of the asset

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[1] Early models can be found in Cootner (ed.) (1964). New developments are collected in VV.AA. (1992).

[2] See Rubinstein (1976) and Leland (1999) for the lognormal case.

exactly as in a restaurant the total bill is the sum of price times quantity of all choices from the menu.
How can we measure the two factors, return and risk, for a given security? Our second assumption states that the relevant quantity of each factor is a marginal quantity: respectively, the marginal increase in return and the marginal increase in risk provided by a marginal unit of the asset added to the market (total) portfolio. Therefore, assets are priced at the margin with respect to their contribution to expected return (mean) and expected risk (variance).

In symbols, over a given time horizon T, let X be a no dividend asset (a random variable representing the asset's cash flow at T) with mean E(X), variance Var(X) and current price P_X . Let M be the market portfolio and P_1 and P_2 be the current prices of the two factors, return and risk.

The current price of a quantity g of the asset X is given by:

$gP_X = P_1$ marginal expected return - P_2 marginal expected risk

where:

marginal expected return = $E(M+gX)- E(M) = g\,E(X)$

marginal expected risk=$Var(M+gX)-Var(M) = g^2Var(X)+2gCov(X,M)$

so that, simplifying:

$P_X = P_1 E(X) - P_2(gVar(X)+2Cov(X,M))$

and letting the quantity g go to 0:

[1.1] $P_X = P_1 E(X) - P_2 2Cov(X,M)$

It is easy to show that this equation for any asset X is the CAPM.

Proposition 1: Equation [1.1] is the CAPM.

Proof: Divide both members by P_X and by P_1 and then subtract 1 and rearrange, obtaining:

$$[1.2] \quad \frac{E(X)}{P_X} - 1 = \frac{1}{P_1} - 1 + 2\frac{P_2}{P_1} P_M \mathrm{Cov}\left(\frac{X}{P_X} - 1, \frac{M}{P_M} - 1\right)$$

where the covariance properties Cov(X+a,M+b)=Cov(X,M) and Cov(X,bM)=bCov(X,M) for constants a and b have been used.

Define the rate of return of the no dividend asset X as the random variable:

$$[1.3] \quad R_X = \frac{X}{P_X} - 1$$

and note that if the asset has a sure, fixed value $X \equiv 1$ at the horizon (i.e. it is a risk free zero coupon discount bond) then Cov(X,M)=0 and:

$$[1.4] \quad P_{RF} = P_1$$

so that the price of the first characteristics is the present value of one unit of money to be received for certain at the future date and the risk free rate is:

$$R_{RF} = \frac{1}{P_{RF}} - 1$$

Substituting in [1.2] we have:

$$[1.5] \quad E(R_X) = R_{RF} + 2\frac{P_2}{P_{RF}} P_M \mathrm{Cov}(R_x, R_M)$$

For the market portfolio:

$$E(R_M) = R_{RF} + 2\frac{P_2}{P_{RF}} P_M \mathrm{Var}(R_M)$$

that is:

$$[1.6] \quad \frac{E(R_M) - R_{RF}}{\mathrm{Var}(R_M)} = 2\frac{P_2}{P_{RF}} P_M$$

to be substituted in [1.5] obtaining:

$$[1.7]\quad E(R_X) = R_{RF} + \frac{Cov(R_x, R_M)}{Var(R_M)}(E(R_M) - R_{RF})$$

Writing $\beta_{X,M} \equiv \dfrac{Cov(R_X, R_M)}{Var(R_M)}$ equation [1.7] is the CAPM in usual form. Q.E.D.

In terms of prices, from [1.1]:

[1.8] $P_M = P_{RF}\, E(M) - 2\, P_2 Var(M)$

and therefore:

$$[1.9]\quad P_X = P_{RF} E(X) - \frac{P_{RF} E(M) - P_M}{Var(M)} Cov(X, M)$$

Equation [1.1] or equivalently [1.9] is the basic valuation equation of any security.

It is interesting to note that, writing, without loss of generality:

[1.10] $E(X) = P_X e^{\rho(T-t)}$ and $1 = P_{RF} e^{r(T-t)}$

for any security equation [1.1] has two representations:

[1.11a] $P_X = e^{-r(T-t)} [P_X\, e^{\rho(T-t)} - P_2 2Cov(X,M)/P_{RF}]$

and

[1.11b] $P_X = e^{-r(T-t)} [P_X\, e^{r(T-t)}] = e^{-r(T-t)}\, \hat{E}(X)$

In the first one, the current price P_X is given by the future expected value E(X), obtained using the natural expected rate of growth ρ, risk-adjusted through the covariance term and discounted at the risk-free rate.
In the second one, the same current price is given by a future expected value $\hat{E}(X)$, simply obtained using the risk-free growth rate r instead of ρ, discounted at the risk-free rate. We say that, in this case, the risk adjustment is not in the process X but in the probabilities (‘risk neutral probabilities’).

This result is a simplified, static version of the equivalent martingale measure theorem of dynamic asset pricing (e.g. Duffie, 1992).

Proposition 2: The valuation equation [1.1] has two equivalent representations:

[1.11a] $P_X = e^{-r(T-t)} E(X- P_2 2(X-E(X))M/P_{RF})$
and
[1.11b] $P_X = e^{-r(T-t)} \hat{\hat{E}}(X)$

where E(.) is the expectation under the natural probability measure and $\hat{\hat{E}}(.)$ is the expectation under the risk-adjusted (risk-neutral) probability measure.

Proof: See above.

2. Pricing options in the mean-variance framework

Let C=max(0, S-K) be the final value of an European call option written on a no dividend asset with future price S, with strike price K and maturity T.
According to [1.1] the price of C is given by:

[2.1] $P_{Call} = P_1 E(C) - P_2 2Cov(C,M)$

and we shall show that, under normality, equation [2.1] is the Black and Scholes option price.
In order to do this, we have first to calculate the Black and Scholes price in the case of normal distributions.

Lemma 1: Let S be a normal variable, $N(\mu_{S,}\sigma_S^2)$, with density n(.). Then:

$$E(C) \equiv E(\max(0, S-K)) = \frac{\sigma_S}{\sqrt{2\pi}} \exp(-\frac{(K-\mu_S)^2}{2\sigma_S^2}) + (\mu_S - K)(1-\Phi(\frac{K-\mu_S}{\sigma_S}))$$

where $\Phi(u)$ is the integral of the standard normal density ϕ up to u.

Proof:
$$E(\max(0,S-K)) = \int_K^{\infty} s n(s) dx - K\int_K^{\infty} n(s) ds$$

then calculate the first integral as:

$$\int_K^{+\infty} s n(s) ds = -\sigma_S^2 \int_K^{+\infty} -\frac{(s-\mu_S+\mu_S)}{\sigma_S^2}\frac{1}{\sqrt{2\pi\sigma_S^2}}\exp\left(-\frac{(s-\mu_S)^2}{2\sigma_S^2}\right) ds =$$
$$-\frac{\sigma_S}{\sqrt{2\pi}}\left[\exp\left(-\frac{(s-\mu_S)^2}{2\sigma_S^2}\right)\right]_K^{+\infty} + \mu_S\left(1-\Phi\left(\frac{K-\mu_S}{\sigma_S}\right)\right)$$

and note that the second one is $1-\Phi\left(\frac{K-\mu_S}{\sigma_S}\right)$. Q.E.D.

Lemma 2: Let S(t) be the Gaussian diffusion process solution of the stochastic differential equation:

$$[2.2] \quad \begin{aligned} dS(t) &= (AS(t)+a)dt + \sigma\, dW(t) \\ S(t_0) &= S_0 \end{aligned}$$

with A,a,σ constant and W(t) standard brownian motion.
Then:

$$[2.3] \quad S(t) = S_0 e^{A(t-t_0)} + \frac{a}{A}(e^{A(t-t_0)}-1) + \sigma \int_{t_0}^{t} e^{A(t-v)} dW(v)$$

and the conditional distribution of S(T) given S(t) is:

$$[2.4] \quad S(T)|S(t) \approx N\left(S(t)e^{A(T-t)} + \frac{a}{A}(e^{A(T-t)}-1), \frac{\sigma^2}{2A}(e^{2A(T-t)}-1)\right)$$

Proof: See Arnold (1974), p.159.

Proposition 3: Let S(t) be the price of a no dividend asset with dynamics:

$$[2.5] \quad dS(t) = \mu(t,S)dt + \sigma\, dW(t)$$

and let C(t, S) be the price of an European call option maturing at time $T \equiv t+\tau$ with strike price K.

Then:

[2.6]

$$C(t,S)=\frac{\exp(-r\tau)\hat{\sigma}_S}{\sqrt{2\pi}}\exp\left(-\frac{(K-S(t)e^{r\tau})^2}{2\hat{\sigma}_S^2}\right)+(S(t)-Ke^{-r\tau})\left(1-\Phi\left(\frac{K-S(t)e^{r\tau}}{\hat{\sigma}_S}\right)\right)$$

$$\hat{\sigma}_S=\sqrt{\frac{\sigma^2(\exp(2r\tau)-1)}{2r}}$$

r being the continuously compounded instantaneous riskless rate.

Proof: Writing C_S for the first partial derivative, $\frac{\partial C(t,S)}{\partial S}$, and C_{SS} for the second partial derivative, the value V of an arbitrage portfolio being long C_S unit of the underlying asset and short one unit of the call is:

[2.7] $V(S, C)=C_S S-C$

By Ito lemma the dynamics of the call price C(S,t) and the arbitrage portfolio (which is linear in S and C) are given by:

[2.8] $dC=C_S dS+C_t dt+½C_{SS}\sigma^2 dt$

[2.9] $dV=V_S dS+V_C dC = C_S dS-dC = -C_t dt-½C_{SS}\sigma^2 dt$

Given that V is instantaneously riskless it must gain the riskless rate:

[2.10] $dV=rVdt=r(C_S S-C)dt$

so that, combining [2.9] and [2.10] we obtain the problem:

$½C_{SS}\sigma^2 + C_S Sr + C_t -Cr = 0$
$dS(t) = \mu(S,t)dt + \sigma dW(t)$
$C(S,T)=\max(0, S(T)-K)$

Writing:

$$dS(t)=Srdt+\sigma\left(\frac{\mu-Sr}{\sigma}dt+dW(t)\right)\equiv Srdt+\sigma dZ(t)$$

where Z(t) is, by Girsanov theorem, a standard brownian motion in a different space, the problem is now the following:

$\frac{1}{2}C_{SS}\sigma^2 + C_S Sr + C_t - Cr = 0$
$dS(t) = Srdt + \sigma dZ(t)$
$C(S,T) = \max(0, S(T)-K)$

whose solution has the stochastic representation in terms of conditional expectations $\hat{\hat{E}}_t$ (Friedman, 1975 p.147) in the probability space induced by Z(t) (risk-neutral probability space):

$$C(S,t) = \hat{\hat{E}}_t(\max(0, S(T) - K)\exp(-r\tau))$$

Noting that by lemma 2 with A=r and a=0:

$$\text{[2.11]} \quad \begin{aligned} &S(T)\big|S(t) \approx N(S(t)\exp(r\tau), \hat{\sigma}_S^2) \\ &\hat{\sigma}_S^2 \equiv \frac{\sigma^2}{2r}(\exp(2r\tau) - 1) \end{aligned}$$

we have, from lemma 1 the required result. Q.E.D.

3. Equivalence of CAPM and option pricing in the normal case.

The same valuation result can be obtained using the CAPM formula [2.1].

Lemma 3: If (S,M) are jointly normal with density:

$$n(s,m) = \frac{1}{2\pi\sigma_S\sigma_M\sqrt{1-\rho_{S,M}^2}}\exp\left(\frac{-1}{2(1-\rho_{S.M}^2)}\left[\left(\frac{s-E(S)}{\sigma_S}\right)^2 - 2\rho_{S.M}\frac{(s-E(S))}{\sigma_S}\frac{(m-E(M))}{\sigma_M} + \left(\frac{m-E(M)}{\sigma_M}\right)^2\right]\right.$$

then the conditional distribution of $M|s$ is given by:

$$M\big|s \approx N\left(E(M) + \rho_{S,M}\frac{\sigma_M}{\sigma_S}(s - E(S)), \sigma_M^2(1-\rho_{S,M}^2)\right)$$

Proof: See Press (1972), p. 69.

Lemma 4: $\mathrm{Cov}(C,M)=\mathrm{Cov}(S,M)(1-\Phi(\frac{K-E(S)}{\sigma_S}))$

Proof:

$$\mathrm{Cov}(C,M)=E[(C-E(C))(M-E(M)]=$$

$$\int_R\int_R[\max(0,S-K)-E(C)](M-E(M))n(s,m)dsdm=$$

$$\int_{-\infty}^{K}\int_R -E(C)(M-E(M))n(s)n(m|s)dsdm+$$

$$\int_K^{+\infty}\int_R(S-K-E(C))(M-E(M))n(s)n(m|s)dsdm=$$

$$\int_{-\infty}^{K}-E(C)[\int_R(M-E(M))n(m|s)dm]n(s)ds+$$

$$\int_K^{+\infty}(s-K-E(C))[\int_R(M-E(M))n(m|s)dm]n(s)ds$$

But from the previous lemma:

$$E(M-E(M)|s)=\rho_{S,M}\frac{\sigma_M}{\sigma_S}(s-E(S))$$

so that:

$$\mathrm{Cov}(C,M)=\int_{-\infty}^{K}-E(C)\rho_{S,M}\frac{\sigma_M}{\sigma_S}(s-E(S))n(s)ds+$$

$$\int_K^{+\infty}(s-K-E(C))\rho_{S,M}\frac{\sigma_M}{\sigma_S}(s-E(S))n(s)ds=$$

$$\rho_{S,M}\frac{\sigma_M}{\sigma_S}\int_K^{+\infty}(s-K)(s-E(S))n(s)ds=$$

$$\rho_{S,M}\frac{\sigma_M}{\sigma_S}[\int_K^{+\infty}(s-E(S))^2n(s)ds+(E(S)-K)\int_K^{+\infty}(s-E(S))n(s)ds]$$

For the first integral:

$$\int_K^{+\infty}(s-E(S))^2 n(s)ds = -\frac{\sigma_S}{\sqrt{2\pi}}\int_K^{+\infty}(s-E(S))\frac{d\exp(-\frac{(s-E(S))^2}{2\sigma_S^2})}{ds}ds =$$

by integration by parts:

$$-\frac{\sigma_S}{\sqrt{2\pi}}[(s-E(S))\exp(-\frac{(s-E(S))^2}{2\sigma_S^2})]_K^{+\infty} + \sigma_S^2(1-\Phi(\frac{K-E(S)}{\sigma_S})) =$$

$$\frac{\sigma_S}{\sqrt{2\pi}}(K-E(S))\exp(-\frac{(K-E(S))^2}{2\sigma_S^2}) + \sigma_S^2(1-\Phi(\frac{K-E(S)}{\sigma_S}))$$

For the second integral:

$$\int_K^{+\infty}(s-E(S))n(s)ds = -\frac{\sigma_S}{\sqrt{2\pi}}\int_K^{+\infty}\frac{d\exp(-\frac{(s-E(S))^2}{2\sigma_S^2})}{ds}ds =$$

$$-\frac{\sigma_S}{\sqrt{2\pi}}[\exp(-\frac{(s-E(S))^2}{2\sigma_S^2})]_K^{+\infty} = \frac{\sigma_S}{\sqrt{2\pi}}\exp(-\frac{(K-E(S))^2}{2\sigma_S^2})$$

so that:

$$\mathrm{Cov}(C,M) = \rho_{S,M}\frac{\sigma_M}{\sigma_S}[\frac{\sigma_S}{\sqrt{2\pi}}(K-E(S))\exp(-\frac{(K-E(S))^2}{2\sigma_S^2}) +$$

$$\sigma_S^2(1-\Phi(\frac{K-E(S)}{\sigma_S})) + (E(S)-K)\frac{\sigma_S}{\sqrt{2\pi}}\exp(-\frac{(K-E(S))^2}{2\sigma_S^2})] =$$

$$\rho_{S,M}\sigma_S\sigma_M(1-\Phi(\frac{K-E(S)}{\sigma_S}))$$

Proposition 4: The CAPM option price is the Black and Scholes option price under the natural probability measure.

Proof:

From CAPM:

$$
\begin{aligned}
P_{Call} &= P_{RF}E(C) - P_2 2Cov(C, M) = \\
& P_{RF}[\frac{\sigma_S}{\sqrt{2\pi}}\exp(-\frac{(K - E(S))^2}{2\sigma_S^2}) + (E(S) - K)(1 - \Phi(\frac{K - E(S)}{\sigma_S}))] \\
& - P_2 2Cov(S, M)(1 - \Phi(\frac{K - E(S)}{\sigma_S}))
\end{aligned}
$$

But, from Lemma 1 and Proposition 2:

$$
\begin{aligned}
P_{Call} &= P_{RF}E(C) - P_2 2Cov(C, M) = P_{RF}\hat{\hat{E}}(C) = \\
& P_{RF}\frac{\hat{\sigma}_S}{\sqrt{2\pi}}\exp(-\frac{(K - S(t)e^{r\tau})^2}{2\hat{\sigma}_S^2}) + (S(t) - Ke^{-r\tau})(1 - \Phi(\frac{K - S(t)e^{r\tau}}{\hat{\sigma}_S}))
\end{aligned}
$$

which is the Black and Scholes price [2.6]. Q.E.D.

4. Conclusion

In these years the proliferation of financial asset of many types has been enormous. This paper tries to explore whether the apparent multiplicity of rights and obligations may be tackled through one simple valuation approach. In a Gaussian world asset prices are obtained through the valuation of two basic characteristics, expected value and variance. We have shown that the valuation formula agrees both with the CAPM and the Back and Scholes' no-arbitrage pricing of options. Extension to non-normal distributions is in our research agenda.

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