Dynamic Duopoly with Vertical Di¤erentiation¤

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Abstract

I analyse a di¤erential game where …rms, through capital accumulation over time, supply vertically di¤erentiated goods. This proves that several results obtained by the static approach are not robust. I show that (i) the sustainability of the duopoly regime is conditional upon the level of …rms' R&D investments; (ii) there are quality ranges where the low quality …rm invests more than the high quality …rm; (ii) there are quality ranges where the low quality …rm's pro…ts are larger than the high quality …rm's.

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1 Introduction

I propose a dynamic approach to the strategic use of non-price tools in a di¤erential game model of vertical di¤erentiation.. Non-price variables typically include product and/or process R&D, product di¤erentiation and advertising, that …rms may use in isolation or together, so as to increase the pro…tability of their price or quantity strategies.

Ever since the pioneering work of Spence (1975) and Mussa and Rosen (1978) on the provision of product quality by a monopolist, vertical di¤erentiation has received wide attention within the theory of industrial organization. Several issues have been investigated in oligopoly models where …rms supply goods of di¤erent quality. In Gabszewicz and Thisse (1979, 1980) and Shaked and Sutton (1982, 1983), the so-called …niteness property is established, according to which the number of …rms that can survive in a vertically is …nite. This result holds if unit costs of quality are ‡at enough, and the overall cost associated with the improvement of quality is an R&D cost unrelated with the scale of production. In their approach, the only costs explicitly modelled is a …xed cost which is assumed to be exogenous and arbitrarily small. Therefore, the …niteness property essentially depends on demand rather than technological conditions. The in‡uence of the shape of the cost function on prices, market shares and pro…ts is the topic of several contributions, where the cost of quality is alternatively related or unrelated with the output scale.[1]

More recent contributions deal several aspects of the technology associated with product innovation in vertically di¤erentiated markets, through either independent ventures (Beath et al., 1987; Dutta et al., 1995; van Dijk, 1996; Rosenkranz, 1997) or joint ventures (Motta, 1992; Rosenkranz, 1995; Lambertini, 2000; Lambertini et al., 2000). A result common to all these contributions is that the highest quality good is more pro…table than all inferior varieties, irrespective of the speci…cation of the cost function and, in particular, notwithstanding the assumption, common to all this literature, that the higher the quality of a good, the higher its cost.

With the exception of Beath et al. (1987) and Dutta et al. (1995), modelling quality improvement as an uncertain innovation race, the above literature adopts a static approach where …rms set qualities and prices (or outputs) in two stages. To the best of my knowledge, the problem of quality supply has been investigated in optimal control and di¤erential game models only in relation with advertising strategies aimed at the formation of goodwill

[1] For models where the development of quality bears upon variable costs, see Moorthy (1988); Champsaur and Rochet (1989); Cremer and Thisse (1994); Lambertini (1996). For those where quality represents a …xed cost, see Aoki and Prusa (1997); Lehmann-Grube (1997) and Lambertini (1999). A comparative evaluation is in Motta (1993).

(see Kotowitz and Mathewson, 1979; Conrad, 1985; and Ringbeck, 1985).[2]

I investigate a differential duopoly game where firms supply goods of different quality, which is the result of capital accumulation over time. The setup of market demand is borrowed from well known static models. The introduction of dynamic capital accumulation over time allows me to show that several results of the static approach are not robust. First of all, the sustainability of the duopoly regime depends upon the non-negativity of firms' profits, which in turn depends on the size of their respective investment in R&D to improve quality. Second, the differential game admits quality ranges where the R&D effort of the low quality firm is larger than the high quality firm's. Third, the dynamic model produces situations where the low quality firm earns higher profits than the high quality firm.

The remainder of the paper is organised as follows. Section 2 contains a brief review of the static model. The differential game is described in section 3 and discussed in section 4. Section 5 contains some concluding remarks.

2 Preliminaries: a summary of the static two-stage game

Here I briefly summarise the static two-stage model analysed in several contributions (Choi and Shin, 1992; Dutta et al., 1995; Wauthy, 1996; Lambertini et al., 2000, inter alia).

Two single-product firms, labelled as H and L, supply goods of qualities $Q > q_H \geq q_L \geq 0$. Consumers are uniformly distributed with density equal to one over the interval $[\overline{\theta} - 1, \overline{\theta}]$, with $\overline{\theta} > 1$. Therefore, the total population of consumers is represented by a unit square. Each consumer is indexed by a marginal willingness to pay for quality $\theta \in [\overline{\theta} - 1, \overline{\theta}]$, and his net utility from consumption is:

$$U = \begin{cases} \theta q_i - p_i \geq 0 \text{ if he buys} \\ 0 \text{ if he doesn't buy} \end{cases} \tag{1}$$

where p_i is the price of the good supplied by firm i at time t.

All costs are assumed to be nil, which entails that any quality improvement is costless.[3] Demands for the two goods are:

$$x_H = \overline{\theta} - \theta_H \, ; \; x_L = \theta_H - \theta_L \, ; \tag{2}$$

[2] For exhaustive surveys on dynamic advertising, see Sethi (1977); Jørgensen (1982); Feichtinger and Jørgensen (1983); Erickson (1991); Feichtinger, Hartl and Sethi (1994).

[3] Dutta et al. (1995) and Lambertini et al. (2000) assume that quality improvements require an R&D cost which, however, is not explicitly defined as a function of quality. Therefore, such a cost does not affect first order conditions.

where θ_H is the marginal willingness to pay for quality characterising the consumer who is indifferent between q_H and q_L at the price vector $\{p_H; p_L\}$; i.e., it is the solution to:

$$\theta_H q_H - p_H = \theta_H q_L - p_L \iff \theta_H = \frac{p_H - p_L}{q_H - q_L}; \tag{3}$$

while θ_L is the marginal evaluation of quality associated with the consumer who is indifferent between buying the low quality good and not buying at all:

$$\theta_L = \frac{p_L}{q_L}. \tag{4}$$

Firms' profits, which coincide with revenues, are:

$$\pi_H = p_H \left(\bar{\theta} - \frac{p_H - p_L}{q_H - q_L} \right); \; \pi_L = p_L \left(\frac{p_H - p_L}{q_H - q_L} - \frac{p_L}{q_L} \right). \tag{5}$$

Firms play simultaneously a non-cooperative two-stage game, where they set qualities in the first stage and price in the second. As usual, the solution concept is subgame perfection by backward induction. The outcome is summarised in the following Proposition, the complete proof of which can be found in Choi and Shin (1992) and Wauthy (1996) (superscript sp stands for subgame perfect):

Proposition 1 At the subgame perfect equilibrium,

- qualities are $q_H^{sp} = Q$ and $q_L^{sp} = 4Q/7$;
- output levels are $x_H^{sp} = 7\bar{\theta}/12$ and $x_L^{sp} = 7\bar{\theta}/24 = x_H^{sp}/2$;
- prices are $p_H^{sp} = \bar{\theta}Q/4$ and $p_H^{sp} = \bar{\theta}Q/14$.

This also allows to establish that $x_H^{sp} + x_L^{sp} < 1$ for all $\bar{\theta} < 8/7$. Hence, for all $\bar{\theta} \geq 8/7$, the demand functions (2) are not valid and the model must be respecified with $x_L = \theta_H - (\bar{\theta} - 1)$.[4]

Moreover, $x_H^{sp} = 7\bar{\theta}/12$ implies that $x_L^{sp} > 0$ iff $\bar{\theta} < 12/7$. Therefore, Proposition 1 produces the following Corollary:

Corollary 1 For all $\bar{\theta} \geq 12/7$, the market is monopolised by the high quality firm.

[4] For the analysis if this case, see Tirole (1988, appendix to ch. 7), Rosenkranz (1995) and Wauthy (1996).

Corollary 1 is an instance of the so-called finiteness property (see Gabszewicz and Thisse, 1979, 1980; Shaked and Sutton, 1982, 1983), which establishes that the demand structure of a vertically differentiated market allows for a finite number of firms operating with positive demand and profits at the subgame perfect equilibrium. In particular, the above case is what Shaked and Sutton label as a natural monopoly. They use consumer income, while here I use the marginal evaluation for quality, as in Mussa and Rosen (1978) and in Gabszewicz and Thisse (1979, 1980). It can be easily shown that the two approaches are equivalent, provided that consumer's utility function is concave in income.[5]

For future reference, it is worth noting that the (exogenously imposed) upper bound of the quality spectrum, Q, which is usually referred to as the highest technologically feasible quality, generates the corner solution $q_H^{sp} = Q$, as the revenues of firm H are everywhere increasing in q_H. Being quality improvements costless, without such a boundary q_H would become infinitely high. Moreover, it is reasonable to think that the upper bound Q should be endogenised, in the sense that firms' R&D investments (in particular, firm H's) should increase the feasible quality range.[6]

3 The dynamic model

Let the market exist over $t \in [0, \infty)$. At any t, as in the static version, a constant population of consumers is uniformly distributed with density equal to one over the interval $[\bar{\theta} - 1, \bar{\theta}]$, with $\bar{\theta} \in (1, 4/3)$.[7] Again, the total mass of consumer is 1. Each consumer is characterised by a marginal willingness to pay for quality $\theta \in [\bar{\theta} - 1, \bar{\theta}]$, and his net instantaneous utility from consumption is now defined as:

$$U = \begin{cases} \theta q_i(t) - p_i(t) \geq 0 & \text{if he buys} \\ 0 & \text{if he doesn't buy} \end{cases} \tag{6}$$

where $q_i(t)$ is the quality and $p_i(t)$ is the price of the good supplied by firm i at time t. Two single-product firms, labelled as H and L, supply goods of

[5] Under this condition, $\theta = \beta / u_y$, where $u_y \equiv \partial u(y)/\partial y$ is the marginal utility of income and β is a positive parameter. If $u_{yy} \equiv \partial^2 u(y)/\partial y^2 \leq 0$, the marginal willingness to pay for quality increases as income increases (see Tirole, 1988, ch. 2).

[6] Of course this idea can be developed in a static context by assuming, e.g., a convex cost of quality improvement. This is investigated in Motta (1993), Lambertini (1996, 1999, 2000), and Lehmann-Grube (1997), inter alia. However, in such a case the derivation of the subgame perfect equilibrium requires numerical calculations.

[7] The meaning of the upper bound to $\bar{\theta}$ will become clear in the remainder of the analysis. See below.

qualities $1 > q_H(t) \geq q_L(t) \geq 0.$

The quality of firm i's product increases over time according to the following dynamics:

$$\frac{\partial q_i(t)}{\partial t} = a\sqrt{k_i(t)}\,; \tag{7}$$

where $k_i(t)$ is the instantaneous investment of firm i in an R&D process aimed at improving product quality, and a is a positive parameter. The initial condition for firm i is $q_i(0) = q_{i0} \geq 0.$ The instantaneous cost associated to the R&D activity is $C_i(k_i(t)) = \rho k_i(t)$; i.e., I assume that the rental price of the capital input be equal to the discount rate $\rho.$

Each firm bears no costs other than $C_i(k_i(t)).$ That is, operative production costs are assumed to be nil, and therefore instantaneous profits are given by the difference between revenues and the cost of investment.

The definition of market demands is analogous to the static setup. At any t; market demands for the two varieties are defined as follows:

$$x_H(t) = \bar{\theta} - \theta_H(t)\,;\; x_L(t) = \theta_H(t) - \theta_L(t)\,; \tag{8}$$

where $\theta_H(t)$ is the marginal willingness to pay for quality characterising the consumer who is indifferent between $q_H(t)$ and $q_L(t)$ at the price vector $\{p_H(t); p_L(t)\}$:

$$\theta_H(t) = \frac{p_H(t) - p_L(t)}{q_H(t) - q_L(t)}\,; \tag{9}$$

while $\theta_L(t)$ is:

$$\theta_L(t) = \frac{p_L(t)}{q_L(t)}\,. \tag{10}$$

Accordingly, instantaneous profits are:

$$\pi_H(t) = p_H(t)\left(\bar{\theta} - \frac{p_H(t) - p_L(t)}{q_H(t) - q_L(t)}\right) - \rho k_H(t)\,; \tag{11}$$

$$\pi_L(t) = p_L(t)\left(\frac{p_H(t) - p_L(t)}{q_H(t) - q_L(t)} - \frac{p_L(t)}{q_L(t)}\right) - \rho k_L(t)\,; \tag{12}$$

provided that $x_H(t) + x_L(t) \leq 1.$

Control variables are the price $p_i(t)$ and the R&D effort $k_i(t)$; while quality $q_i(t)$ is the state variable. Firms play simultaneously and non-cooperatively. Given that the dynamic constraint (7) is non-linear, I will confine to the open-loop solution concept. Although the open-loop solution is only weakly time consistent, in some circumstances it can be justified by considering that it may be extremely costly for firms to change long-run investment plans at any intermediate date.[8] Firm i's Hamiltonian is:

$$H_i(t) = e^{-\rho t} \cdot \pi_i(t) + \lambda_i(t)\left[a\sqrt{k_i(t)}\right] ; \tag{13}$$

where $\lambda_i(t) = \mu_i(t)e^{\rho t}$; and $\mu_i(t)$ is the co-state variable associated to $q_i(t)$. The optimality conditions are (henceforth, for the sake of brevity, I will drop the indication of time):

$$\frac{\partial H_H}{\partial p_H} = \frac{p_L - 2p_H + \theta\,(q_H - q_L)}{q_H - q_L} = 0 ; \tag{14}$$

$$\frac{\partial H_L}{\partial p_L} = \frac{p_H q_L - 2p_L q_H}{q_L\,(q_H - q_L)} = 0 ; \tag{15}$$

$$\frac{\partial H_i}{\partial k_i} = -\rho + \frac{a\lambda_i}{2\sqrt{k_i}} = 0 ; \; i = H; L ; \tag{16}$$

$$-\frac{\partial H_H}{\partial q_H} = \frac{\partial \mu_H}{\partial t} \Rightarrow \frac{\partial \lambda_H}{\partial t} = \rho\lambda_H - \frac{p_H\,(p_H - p_L)}{(q_H - q_L)^2} ; \tag{17}$$

$$-\frac{\partial H_L}{\partial q_L} = \frac{\partial \mu_L}{\partial t} \Rightarrow \frac{\partial \lambda_L}{\partial t} = \rho\lambda_L - \frac{p_L\,(p_H q_L^2 - 2p_L q_H q_L + p_L q_H^2)}{[q_L\,(q_H - q_L)]^2} ; \tag{18}$$

$$\lim_{t \to \infty} \mu_i(t) \cdot q_i(t) = 0 ; \; i = H; L . \tag{19}$$

Solving (14-15) yields optimal prices:

$$p_H^* = \frac{2\theta q_H\,(q_H - q_L)}{(4q_H - q_L)} ; \; p_L^* = \frac{\theta q_L\,(q_H - q_L)}{(4q_H - q_L)} ; \tag{20}$$

with $p_H^* > p_L^* > 0$ for all $q_H > q_L$. If $q_H = q_L$, then $p_H^* = p_L^* = 0$ and the allocation of market demand across firms is not determined. In such a case however, independently of the allocation of costumers, revenues are zero and therefore profits are negative for both firms. It is worth noting that the above expressions coincide with the solutions of the price stage of the static game (see section 2).

[8] Indeed, the main difference between the open-loop solution and the closed-loop and feedback ones is that in the former, players decide by looking at the clock (i.e., calendar time), while in the latter, they decide by looking at the stock (i.e., the past history of the game). Whether the second picture is more realistic than the first has to be evaluated within the specific model being used, in connection with the kind of story the model itself tries to account for (Clemhout and Wan, 1994, p. 812). See also Başar and Olsder (1982, 1995[2]), in particular ch. 6.

Now, from (16) I obtain:

$$\lambda_i = \frac{2\rho\sqrt{k_i}}{a} \tag{21}$$

and

$$k_i = \left(\frac{a\lambda_i}{2\rho}\right)^2 \tag{22}$$

which yields the dynamics of firm i's R&D effort:

$$\frac{\partial k_i}{\partial t} = \frac{a^2\lambda_i}{2\rho^2} \cdot \frac{\partial \lambda_i}{\partial t}. \tag{23}$$

Then, I can rewrite (23) for both firms, using (17), (18), (20) and (21), to obtain:

$$\frac{\partial k_H}{\partial t} \propto \sqrt{k_H}\left[\rho^2\sqrt{k_H}\left(4q_H - q_L\right)^2 - a\theta^2 q_H\left(2q_H - q_L\right)\right]; \tag{24}$$

$$\frac{\partial k_L}{\partial t} \propto \sqrt{k_L}\left[2\rho^2\sqrt{k_L}\left(4q_H - q_L\right)^2 - a\theta^2 q_H^2\right]. \tag{25}$$

Therefore, $\{\partial k_H/\partial t = 0; \partial k_L/\partial t = 0\}$ when either $k_H = k_L = 0$, in which case $q_i = q_{i0}$ forever, or

$$k_H = \frac{a^2\theta^4 q_H^2\left(2q_H - q_L\right)^2}{\rho^4\left(4q_H - q_L\right)^4}; \; k_L = \frac{a^2\theta^4 q_H^4}{4\rho^4\left(4q_H - q_L\right)^4}. \tag{26}$$

In the remainder I shall focus upon (26), which depicts the economically relevant situation. Capital levels (26) imply the following result:

Lemma 1 For all

$$q_H \in \left(q_L; \frac{2q_L + \sqrt{4q_L^2 + 1}}{4}\right);$$

we have $k_H < k_L$. For all

$$q_H \geq \frac{2q_L + \sqrt{4q_L^2 + 1}}{4};$$

we have $k_H \geq k_L$.

Proof. To prove the Lemma, observe first that

$$k_H - k_L = \frac{a^2 \Theta^4 q_H^2}{4\rho^4 (4q_H - q_L)^4} \cdot \left(16q_H^2 + 4q_L^2 - 16q_H q_L\right) = 0 \qquad (27)$$

at

$$q_H = \frac{2q_L \pm \sqrt{4q_L^2 + 1}}{4},$$

with the smaller root being always negative, and the larger root being higher than q_L. Then, consider that the coefficient of q_H^2 in (27) is positive. This concludes the proof. ■

In contrast with the static literature on R&D investment aimed at increasing product quality (see, e.g., Dutta et al. 1995; Lehmann-Grube, 1997; Lambertini et al., 2000), in the dynamic setup adopted here it is not generally true that the high quality firm carries out a higher effort than the rival in improving product quality.

Now consider qualities. According to (7), qualities seems to be always increasing over time. Thus, the flow of investments carried out by the high quality firm keeps pushing the upwards the top feasible quality over time. However, under the specification of the demand function as in (8), there exists an upper bound to the quality ratio q_L/q_H. To verify this, observe that plugging (20) into x_L yields the following expression:

$$x_L = \frac{\Theta q_H}{(4q_H - q_L)} \qquad (28)$$

which is increasing in q_L. When

$$q_L = (4 - 3\Theta) q_H \in (0, q_H) \text{ for all } \Theta \in \left(1, \frac{4}{3}\right), \qquad (29)$$

we have $\theta_L = \Theta - 1$ and $x_H + x_L = 1$; i.e., the market is fully covered.[9] Moreover, also q_H is increasing in q_L. The intuition behind this phenomenon is that any increase in the low quality enhances total demand, because it increases the surplus of low-income consumers. This, in turn, causes firm H to increase the quality of her own variety as well, by strategic complementarity, and this generates a larger demand for the high quality good. In particular, $x_H = 1$ and $x_L = 0$ for all $q_L \geq 2(2 - \Theta) q_H$, provided

[9] For all $q_L \geq (4 - 3\Theta) q_H$ or $\Theta \geq 4/3$, the market is completely covered and the demand functions (8) are no longer valid. The model must be reformulated taking into account that $x_H = \Theta - \theta_H$ and $x_L = \theta_H - \Theta + 1$.

$\theta \in (1; 2)$: However, $2(2 - \theta) > (4 - 3\theta)$: Therefore, as long as the market is only partially covered, or at most is fully covered at the margin with $q_L = (4 - 3\theta) q_H$; the market demand for the low quality good is strictly positive. At $q_L = (4 - 3\theta) q_H$; we have:

$$\begin{aligned} x_H^* &= \frac{2}{3}\,;\; x_L^* = \frac{1}{3}\,; \\ p_H^* &= 2(\theta - 1) q_H\,;\; p_L^* = (4 - 3\theta)(\theta - 1) q_H\,; \\ k_H^* &= \frac{a^2 (3\theta - 2)^2}{81\rho^4}\,;\; k_L^* = \frac{a^2}{324\rho^4}\,: \end{aligned} \tag{30}$$

where $k_H^* > k_L^*$: Under full market coverage, instantaneous profits amount to:

$$\begin{aligned} \pi_H^* &= \frac{108\rho^3 (\theta - 1) q_H - a^2 (3\theta - 2)^2}{81\rho^3}\,; \\ \pi_L^* &= \frac{81\rho^3 (7\theta - 3\theta^2 - 4) q_H - a^2}{324\rho^3}\,: \end{aligned} \tag{31}$$

4 Discussion

Here I would like to discuss some qualitative properties of the dynamic model, as well as the sustainability of duopoly at $q_L / q_H = (4 - 3\theta)$; i.e., when full market coverage obtains at the margin. A further issue consists in the comparative assessment of the dynamic model vs the static one.

First of all, recall from Lemma 1 that

$$k_H^* \geq k_L^* \text{ iff } q_H \geq \frac{2q_L + \sqrt{4q_L^2 + 1}}{4}\,: \tag{32}$$

For (32) to hold at $q_L / q_H = (4 - 3\theta)$; the following inequality must be satisfied:

$$\frac{q_L}{4 - 3\theta} \geq \frac{2q_L + \sqrt{4q_L^2 + 1}}{4}\,;$$

which is true iff

$$q_L \geq \frac{4 - 3\theta}{4\sqrt{\theta - 1}} > 0 \text{ for all } \theta \in \left(1; \frac{4}{3}\right)\,:$$

This produces the following result:

Proposition 2 For all $q_L \in \left(0; \frac{4 - 3\theta}{4\sqrt{\theta - 1}}\right)$; we have that

$$q_H = \frac{q_L}{4 - 3\theta} < \frac{2q_L + \sqrt{4q_L^2 + 1}}{4} \;\Rightarrow\; k_H^* < k_L^*\,:$$

For all $q_L \geq \frac{4-3\theta}{4\sqrt{\theta-1}}$; we have that

$$q_H = \frac{q_L}{4-3\theta} \geq \frac{2q_L + \sqrt{4q_L^2+1}}{4} \Rightarrow k_H^* \geq k_L^*.$$

In words, Proposition 2 says that, contrary to the well known view adopted in the static approach to vertical differentiation, a situation where the high quality firm invests less than the rival in quality-improving activities is indeed admissible. As long as q_L is sufficiently low, there is no incentive for firm H to produce a larger R&D effort than firm L does.

Now consider the sustainability of the duopoly regime. This relates to the finiteness property (Shaked and Sutton, 1983) according to which the number of firms that can survive in a vertically differentiated market with positive demands, prices above marginal costs and positive profits is finite. Using a static approach, Shaked and Sutton derive this property on the basis of market affluence, under the assumption of arbitrarily small fixed (exogenous) costs. If so, the condition such that, e.g., $x_H = 1$ can be easily established. Accordingly, the market is labelled as a natural monopoly in that there exists no demand for a low quality good. The same procedure can be used to derive condition for a natural oligopoly with any number of firms.

However, in the dynamic setting there are non-negligible costs associated to the development of product quality, and therefore the sustainability of either the monopoly or the duopoly regime depends upon the non-negativity of profits (31).

The positivity of π_L^* requires the following condition to be met:

$$q_H > \frac{a^2}{108\rho^2(7\theta - 3\theta^2 - 4)} \equiv \tilde{q}_H > 0 \text{ for all } \theta \in \left(1; \frac{4}{3}\right). \tag{33}$$

The profits of the high quality firm are positive iff:

$$q_H > \frac{a^2(3\theta - 2)^2}{108\rho^2(\theta - 1)} \equiv \hat{q}_H > 0 \text{ for all } \theta \in \left(1; \frac{4}{3}\right). \tag{34}$$

The following is a relevant complement:

$$\pi_H^* > \pi_L^* \text{ iff } q_H > \frac{a^2(6\theta - 5)(2\theta - 1)}{108\rho^2(\theta - 1)} \equiv \bar{q}_H > 0 \text{ for all } \theta \in \left(1; \frac{4}{3}\right). \tag{35}$$

It is quickly established that

$$
\begin{aligned}
\bar{q}_H > \hat{q}_H > \tilde{q}_H > 0 \text{ for all } \theta \in \left(1, \frac{5+\sqrt{5}}{6}\right); \\
0 < \bar{q}_H < \hat{q}_H < \tilde{q}_H \text{ for all } \theta \in \left(\frac{5+\sqrt{5}}{6}, \frac{4}{3}\right).
\end{aligned}
\tag{36}
$$

This allows me to formulate the following Theorem:

Theorem 1 Suppose $q_L = (4 - 3\theta)\, q_H$, so that the market is fully covered at the margin. If so, then:

A] For all $\theta \in \left(1, \frac{5+\sqrt{5}}{6}\right)$, we have $\bar{q}_H > \hat{q}_H > \tilde{q}_H > 0$ and

1. no firm supplies the market if $q_H \in (0, \tilde{q}_H)$;
2. the market is a duopoly with firm H operating at negative profits if $q_H \in (\tilde{q}_H, \hat{q}_H)$, because $\pi_L^* > 0 > \pi_H^*$ in this interval;
3. the market is a duopoly if $q_H \in (\hat{q}_H, \bar{q}_H)$, with $\pi_L^* > \pi_H^* > 0$;
4. the market is a duopoly if $q_H > \bar{q}_H$, with $\pi_H^* > \pi_L^* > 0$.

B] For all $\theta \in \left(\frac{5+\sqrt{5}}{6}, \frac{4}{3}\right)$, we have $0 < \bar{q}_H < \hat{q}_H < \tilde{q}_H$ and

1. no firm supplies the market if $q_H \in (0, \hat{q}_H)$;
2. firm H is a monopolist if $q_H \in (\hat{q}_H, \tilde{q}_H)$, because $\pi_H^* > 0 > \pi_L^*$ in this interval;
3. the market is a duopoly if $q_H > \tilde{q}_H$, with $\pi_H^* > \pi_L^* > 0$.

As q_H keeps increasing over time, regimes A.1,2,3,4 or B.1,2,3 are alternatively to be observed as time goes by, depending upon the level of θ. From an economic standpoint, the most interesting feature emerging from Theorem 1 is that, if the market is not sufficiently rich (i.e., in case A), then the low quality good is more profitable than the high quality good, as long as $q_H < \bar{q}_H$. This strongly contrasts with the results derived by the static analysis, where the profits of the high quality firm are always larger than the profits of the low-quality firm. In the present setting, firm H always earns positive profits if and only $q_{H0} > \hat{q}_H$.

Finally, comparing the results concerning the dynamic model against those derived from the static one (see section 2), there emerges no internal optimum for the high quality level in both models. However, while the upper bound Q of the interval of technologically feasible quality is superimposed to the static model, the dynamic model shows that the top quality level increases over time as firm H keeps investing in R&D activities.

Moreover, the two models yields qualitatively equivalent predictions in terms of the quality ratio (i.e., $q_L/q_H = 4/7$), prices, quantities and profits when $\theta = 8/7 < 4/3$. The admissible range of θ for the duopoly equilibrium to hold under partial market coverage is larger in the differential (open-loop) game that in the static (two-stage) game. In particular, a straightforward comparison between $q_L^N = 4Q/7$ and $q_L = (4 - 3\theta)\, q_H$ yields the following final result:

Proposition 3 For all $\theta \in \left(1, \frac{8}{7}\right)$, the quality ratio $\frac{q_L}{q_H}$ is higher in the differential game than in the static game.

5 Concluding remarks

I have analysed a differential game where firms, through capital accumulation over time, supply vertically differentiated goods. The explicit treatment of R&D activity as a capital accumulation process proves that several results seemingly well established in the static approach are not robust. In particular, I have shown three main results: (i) the sustainability of the duopoly regime is conditional upon the level of firms' R&D investments; (ii) there are quality ranges where the low quality firm invests more than the high quality firm; (ii) there exist quality ranges where the low quality firm's profits are larger than the high quality firm's.

In consideration of the large number of economically relevant issues associated with the supply of product quality in competitive environments, the foregoing analysis represent a preliminary step. Many extensions, e.g., dynamic investment in demand-increasing advertising or the accumulation of capacity for production, are left for future research.

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