Differential Games and Oligopoly Theory: An Overview

Roberto Cellini* - Luca Lambertini**
* Istituto di Economia Politica
Università di Catania
Corso Italia 55, 95129 Catania, Italy
fax +39-095-370574
e-mail cellini@mbox.unict.it
** Dipartimento di Scienze Economiche
Università di Bologna
Strada Maggiore 45, 40125 Bologna, Italy
fax +39-051-2092664
e-mail lamberti@spbo.unibo.it

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Abstract

We illustrate the foundations of the theory of differential games, with particular regard to the definition of information sets and solution concepts. Then, we provide a survey of several relevant applications of the theory to modelling the behaviour of oligopolistic firms.

1 Introduction

This chapter aims at introducing the reader to the dynamic models of oligopolistic competition. In particular, we want to outline the basics of the theory of differential games and provide the reader with a brief survey of the literature concerning its applications to industrial organization. It is surprising that the most part of standard microeconomic analysis - and specifically the theory of industrial organization - has been developed in static contexts, although this is clearly at odds with reality. Even the issue of strategic interaction among firms over time has been modelled mostly through the tools of repeated games, which are inherently static.

The theory of differential games originated from the work of Isaacs (1954), in the form of unpublished reports of the RAND Corporation, accounting for his research activity in the previous five years, at least. The reason why differential game theory remained for a long time at the margin of research in economics is certainly to be found in the fact that Isaacs, as well as many of his colleagues working in the same field or in related fields, was in fact appointed by the US Government to deal with military problems related to the Cold War.[1] Much the same can be told about their Russian counterparts. In both cases, the products of research started being published in the mid-sixties (Isaacs, 1965; Pontryagin, 1966), and, as a result of this delay, their applications to economics are extremely recent and relatively few.

Most of the applications of differential game theory are to be found in the field of industrial organization,[2] and, more precisely, they can be partitioned into four groups:

I. Oligopoly games with dynamic prices

II. Oligopoly games with capital accumulation for production

[1] This is the case for Arrow, Bellman, Nash, von Neumann, Tucker and many others. A very enjoyable account of the activity at RAND Corporation in the early fifties can be found in Nasar (1998). Relevant applications of differential game theory to military issues include Brito (1972), Taylor (1978), Intriligator and Brito (1984, 1989).

[2] Several applications can also be found in macroeconomics. See Pau (1975), Başar, Turnovsky and d'Orey (1986), Pohjola (1986), Başar and Salmon (1988), van der Ploeg and de Zeeuw (1989), de Zeeuw and van der Ploeg (1991).

III. R&D games

IV. Advertising games

In this survey, we give an account of I-III.[3] The paper is organised as follows. First, the foundations of differential games are laid out, together with the Hamiltonian solution method (section 2). Then, we introduce the simplest way to treat dynamics in a market game, reviewing games with dynamic prices where firms bear solely variable costs, i.e., there is no capital accumulation of any kind (section 3). The following step consists in describing both Cournot and Bertrand competition with capital accumulation for production (section 4). Finally, we survey games of innovation, where investment is aimed at achieving either process or product innovation (section 5). Concluding comments are in Section 6.

2 Technical features

Here, we briefly illustrate the cornerstones of the theory of differential games, namely, the notions of

- state variable and control variable
- objective functions of players
- information and related solution concepts

2.1 The state variable and the control variable

In any dynamic settings, at least one variable changes over time, depending on its past values as well as the players' choices. We define this variable as the *state variable*. An example pertaining to industrial organization may be a firm's productive capacity or installed capital, which depend upon both the capacity held by the firm in past periods and her current investment decisions. Insofar as there exists strategic interdependence among firms, both the optimal investment at any point in time, and the resulting evolution of capacity over time, depend upon the investment undertaken by all other

[3] Jørgensen (1982) provides a survey of differential games with advertising. See also Leitmann and Schmitendorf (1978); Feichtinger (1983); Dockner and Feichtinger (1986).

firms. The actions of players at any time t consists in setting the so-called *control variables*. In the jargon of the previous example, current investment is the control variable of the generic firm i, who must set it optimally over time.

Let the game unravel over $t \in [0, T]$.[4] Define the set of players as $\mathbb{P} \equiv \{1, 2, 3, ...N\}$. Moreover, let $x_i(t)$ define the state variable for player i.[5] Formally, its dynamics can be described by the following:

$$\frac{dx_i(t)}{dt} \equiv \dot{x}_i (t) = f\left(x_i(t), \{u_i(t)\}_{i=1}^{N}\right) \quad (1)$$

where $\{u_i(t)\}_{i=1}^{N}$ is the vector of players' actions at time t, i.e., it is the vector of the values of control variables at time t.

The value of the state variables at the beginning of time ($t = 0$) is assumed to be known: $\{x_i(0)\}_{i=1}^{N} = \{x_{0,i}\}_{i=1}^{N}$. The behaviour of the state variable over time represents a dynamic constraint for each player. As long as the state variable affects each player's optimal decisions, and there exists a feedback from the players' actions to the value of each state variable, strategic interdependence among players emerges.

2.2 The objective function

Each player has an objective function, to be either maximised or minimised, depending upon the way we define payoff functions (i.e., whether payoffs denote gains or losses). The function is defined as the discounted value of the flow of payoffs over time. Define the instantaneous payoff accruing to player i at time t as $\pi_i(t)$, and, for the sake of simplicity, suppose $\pi_i(t)$ is a gain (or profit). Of course, the instantaneous payoff must depend upon the choices made by player i as well as its rivals, that is, $\pi_i(t) = \pi_i(x_i(t), x_{-i}(t), u_i(t), u_{-i}(t))$, where $u_{-i}(t)$ summarises the actions of all other players at time t. Player i's objective is then

$$\max_{u_i(t)} J_i \equiv \int_0^T \pi_i(t)e^{-\rho t}dt \quad (2)$$

[4]The time horizon of the game may well be infinitely long. See sections 4 and 5 for models where $t \in [0, \infty)$.

[5]The state variable might be unique for all players, but it is not necessarily so. In the above example, it is certainly not, because the accumulatrion of capacity or capital characterise every individual firm in the market, possibly in different ways. See section 5 for cases where $x(t)$ is indeed unique for all players.

where the factor $e^{-\rho t}$ discounts future gains. Observe that the discount rate ρ has no index, due to the simplifying assumption that all payers discount future payoffs at the same constant rate. In order to solve his optimum problem, each player sets the value of his control variable $u_i(t)$ in each period, so that he actually chooses a time path for his control, under the dynamic constraint represented by the behaviour of the state variable (1).

2.3 Information

What is the relevant information set available to each player at any date $t \in [0, T]$? Dynamic game theory distinguishes three cases:

Open-Loop Information (OLI) Common knowledge consists only in the state of the world, i.e., the vector of values of the state variables, at initial time $t = 0$. At this date, each player sets the path of his control variable (taking into account the expected behaviour of all other players). All decisions are taken at $t = 0$, and applied accordingly by players during the whole relevant time span.

Feedback Information (FI) Players are assumed to know, at any t, the state of the world at $t - 1$, so that the information set at time t can be summarised by the vector of values of the state variables of all players at $t - 1$, defined as $X(t-1) \equiv \{x_1(t-1), x_2(t-1), ... x_N(t-1)\}$ (or $X(t-\varepsilon)$, where ε is positive and arbitrarily small, if the game is specified in continuous time).

Closed-Loop Information (CLI) Under closed-loop (or history-dependent) information, players are assumed to know at date t the whole previous history of the game over $[0, t)$.

In the remainder, we will illustrate industrial organization models under OLI. This solution is weakly time consistent, in the sense that, if one considers the game over the truncated interval $[\sigma, T]$, where $\sigma \in (0, T)$, its solution coincides with the solution to the original game over the same interval, provided that agents have played optimally over $[0, \sigma)$.[6]

[6] The limitations affecting open loop solutions are well known (Kydland, 1977; Spence, 1979; see also Fudenberg and Tirole, 1991, pp. 520-36). In line of principle, it would be preferable to solve a differential game under either FI or, even better, CLI, rather than

2.4 Equilibrium concepts

Exactly like in static games, we may describe strategic interaction between players in different ways. First of all, we may suppose that each player takes all his opponents' choices as give, in which case the relevant equilibrium concept, i.e., the Nash equilibrium, can be defined as usual: a set of strategy paths is a Nash equilibrium if each player consider his own action as optimal given the other players' behaviour, and even after having observed such behaviour.

Second, we may consider the Stackelberg equilibrium, where the leader takes into account the follower's best reply, so that the reaction function of the follower must be inserted into the leader's problem as an additional constraint.

Third, players can cooperate, i.e., they can adopt a common objective function defined, for instance, by the sum of individual discounted flows of payoffs. In the field of industrial organization, this is the case when, e.g., firms build up a cartel in order to maximise joint profits w.r.t. their investment in R&D to reduce marginal production costs or to introduce new products.

2.5 Optimization over time

Solving a differential game amounts to solving a problem of dynamic planning with several agents interacting strategically with each other. We are not going into the formal details of dynamic optimization;[7] rather, we confine to reporting some operative rules to solve a differential game. Namely, we present the Hamilton technique.

Consider the following problem for player i :[8]

$$\max_{u_i(t)} J_i \equiv \int_0^T \pi_i \left(x_i(t), x_{-i}(t), u_i(t), u_{-i}(t) \right) e^{-\rho t} dt \tag{3}$$

under OLI. It can be shown that there are classes of games for which the open-loop and the closed-loop solutions coincide (see Reinganum, 1982b, Mehlmann and Willing ,1983, and Fershtman, 1987). For an exhaustive discussion of time consistency and subgame perfectness in differential game theory, see Mehlmann (1988, ch. 4) and Başar and Olsder (1995^2, chs. 5 and 6).

[7]We refer the reader interested in a thorough exposition of methods for dynamic optimization and differential games to Chiang (1992); and Mehlmann (1988) or Başar and Olsder (1995^2), respectively.

[8]Note that player i may face either a maximization or a minimization problem. In the remainder, we will focus on the former case.

$$s.t.\ \frac{dx_i(t)}{dt} \equiv \dot{x}_i\ (t) = f\left(x_i(t), \{u_i(t)\}_{i=1}^{N}\right) \tag{4}$$

where $x_i(t)$ and $u_i(t)$ denote player i's state variable and control variable, respectively. We introduce now the Hamiltonian function, defined as follows:

$$\mathcal{H}\left(x_i(t), u_i(t)\right) \equiv \left[\pi_i\left(x_i(t), u_i(t), u_{-i}(t)\right) + \lambda_i(t) \cdot f\left(x_i(t), \{u_i(t)\}_{i=1}^{N}\right)\right] \cdot e^{-\rho t}, \tag{5}$$

where $\lambda_i(t) = \mu_i(t)e^{\rho t}$ is an auxiliary variable, called the *co-state variable*, its interpretation being much the same as that attached to Lagrange multipliers in static constrained optimization problems. That is, the co-state variable can be seen as the shadow price of a variation of the state variable.

The first order conditions (FOCs) for the solution of the dynamic problem are:

$$\frac{\partial \mathcal{H}\left(x_i(t), u_i(t)\right)}{\partial u_i(t)} = 0; \tag{6}$$

and

$$-\frac{\partial \mathcal{H}\left(x_i(t), u_i(t)\right)}{\partial x_i(t)} = \frac{\partial \lambda_i(t)}{\partial t} \tag{7}$$

along with the initial condition $x_i(0) = x_0$ and a transversality condition, which sets the final value (at time T) of the state and/or co-state variables. In problems defined over an infinite time horizon, it is very common to set

$$\lim_{t\to\infty} \lambda_i(t) \cdot x_i(t) = 0 \tag{8}$$

as the transversality condition. It amounts to saying that the "monetary" value of the state variable at infinity is nil.

In analysing dynamic settings, we are also generally interested in evaluating whether a *steady state* exists, i.e., a vector of variables which possesses the desirable property that, whenever players reached the steady state, then all the relevant variables would remain unchanged thereafter.

A steady state equilibrium may not exist, or, if it does, it may not be unique.[9] Last but not least, a steady state equilibrium may exhibit different features as far as its stability is concerned. More precisely, the steady state equilibrium can be:

[9] There exists also the possibility that a steady state be meaningless from an economic standpoint. See below, sections 4 and 5.

A. *a stable (unstable) node*, when the system non-cyclically converges to (diverges from) that steady state, regardless of where it starts from;

B. *stable along a saddle path*, when there exists one and only one time path leading to the steady state;

C. *a stable (unstable) focus*, when the system cyclically converges to (diverges from) from the steady state;

D. *a vortex*, when the system orbits around the steady state in a perpetual motion.

Define the steady state as the vector $\{x^*, u^*\}$. This vector is the outcome of the dynamic system:

$$\begin{cases} \frac{dx(t)}{dt} \equiv \dot{x}(t) = f(x,u) = 0 \\ \frac{du(t)}{dt} \equiv \dot{u}(t) = g(x,u) = 0 \end{cases} \tag{9}$$

The dynamic equations in (9) can be linearised around the steady state point through a first order Taylor expansion, so that the system (9) can be written in matrix notation as follows:

$$\begin{bmatrix} \dot{x} \\ \\ \dot{u} \end{bmatrix} = \Xi \begin{bmatrix} (x - x^*) \\ \\ (u - u^*) \end{bmatrix} + \Psi \tag{10}$$

where Ξ is the matrix of partial derivatives evaluated at $\{x^*, u^*\}$:

$$\Xi = \begin{bmatrix} f_x & f_u \\ g_x & g_u \end{bmatrix}_{|x^*,u^*}$$

and $\Psi = \{f(x^*, u^*), g(x^*, u^*)\}$ is a column vector whose components are zero, since $f(\cdot, \cdot)$ and $g(\cdot, \cdot)$ are nil when evaluated at $\{x^*, u^*\}$.[10] The stability properties of the system in the neighbourhood of the steady state depend upon the trace and determinant of matrix Ξ. In particular, the system produces a saddle when the determinant is negative. Of course, in looking for steady

[10] Notice that we have dropped index i. This is admissible if players are symmetric, so that the state and control variables are symmetric in equilibrium.

states, we have to ascertain whether optimality conditions (6-7) are indeed compatible with $dx(t)/dt = 0$ and $du(t)/dt = 0$.

As a last remark on steady state Nash equilibria, observe that the analysis of the properties of a dynamic system is conceptually distinct from and independent of the issue of the equilibrium of a differential game. We have a Nash equilibrium when each agent plays the best response the all his opponents' actions. From the standpoint of the analysis of a dynamic system, "equilibrium" means that variables are stationary over time. Both issues are relevant when we focus upon a steady state Nash equilibrium, i.e., a state where the system (the market, if we refer to industrial organization examples) stays, provided each agent plays his optimal strategy.

We are now in a position to proceed to a selected overview of the existing literature on dynamic oligopoly games. In the next section, we expose a model where firms produce without capital, with variable costs only, and dynamics enters the picture through the evolution of market price over time. Then, in the following sections, we focus upon the dynamics of capital accumulation over time, aimed either at producing final consumption goods, or at achieving process or product innovations through R&D activities.

3 Dynamic prices

Probably, the simplest way to think about the dynamics of market interaction consists in assuming that prices evolve over time according to some acceptable rules. That is, it consists in taking price as the state variable. This is the problem analysed in Simaan and Takayama (1978) and Fershtman and Kamien (1987).[11] In this section, we present a simplified version of the model.

Consider an oligopoly where, at any $t \in [0, \infty)$, N firms produce quantities $q_i(t)$, $i \in \{1, 2, ...N\}$, of the same homogeneous good at a total cost $C_i(t) = cq_i(t) - \frac{1}{2}[q_i(t)]^2$. In each period, market demand is $\widehat{p}(t) = A - B\sum_{i=1}^{N} q_i(t)$. Hence, the problem of firm i is:

$$\max_{q_i(t)} J_i = \int_0^\infty e^{-\rho t}\, q_i(t) \cdot \left[p(t) - c - \frac{1}{2}q_i(t)\right] dt \tag{11}$$

subject to:

$$\frac{dp(t)}{dt} \equiv \dot{p}\,(t) = w\left\{\widehat{p}(t) - p(t)\right\} \tag{12}$$

[11] See also Mehlmann (1988, ch. 5) for an exhaustive exposition of both contributions.

$$p(0) = 0; \text{ and } p(t) \geq 0 \text{ for all } t \in [0, T] . \tag{13}$$

Notice that the dynamics described by (12) establishes that price adjusts proportionately to the difference between the price level given by the inverse demand function and the current price level, the speed of adjustment being determined by the constant w. This amounts to saying that the price mechanism is sticky, that is, firms face menu costs in adjusting their price to the demand conditions deriving from consumers' preferences: they may not (and, in general, the will not) choose outputs so that the price reaches immediately the "correct" market clearing level, given by $\widehat{p}(t)$. The Hamiltonian function is:

$$\mathcal{H}(t) = e^{-\rho t} \cdot \left\{ q_i(t) \cdot \left[p(t) - c - \frac{1}{2} q_i(t) \right] + \lambda_i(t) w \left[A - B \sum_{i=1}^{N} q_i(t) - p(t) \right] \right\} , \tag{14}$$

where $\lambda_i(t) = \mu_i(t) e^{\rho t}$, and $\mu_i(t)$ is the co-state variable associated to $p(t)$. The supplementary variable $\lambda_i(t)$ is introduced to ease calculations as well as the remainder of the exposition. Consider the first order condition (FOC) w.r.t. $q_i(t)$, calculated using (14):

$$\frac{\partial \mathcal{H}(t)}{\partial q_i(t)} = p(t) - c - q_i(t) - \lambda_i(t) B w = 0 . \tag{15}$$

This yields the optimal open-loop output for firm i, as follows:[12]

$$q_i(t) = \begin{cases} p(t) - c - \lambda_i(t) B w & \text{if } p(t) > c + \lambda_i(t) B w \\ 0 & \text{otherwise.} \end{cases} \tag{16}$$

The remaining conditions for optimum are:

$$-\frac{\partial \mathcal{H}(t)}{\partial p(t)} = -q_i(t) + \lambda_i(t) w = \frac{\partial \mu_i(t)}{\partial t} \Rightarrow \frac{\partial \lambda_i(t)}{\partial t} = \lambda_i(t)(w + \rho) - q_i(t) ; \tag{17}$$

$$\lim_{t \to \infty} \mu_i(t) \cdot p(t) = 0 . \tag{18}$$

Differentiating (16) and using (17), we obtain:

$$\frac{d q_i(t)}{dt} = \frac{dp}{dt} - Bw \left[(\rho + w) \lambda_i(t) - q_i(t) \right] . \tag{19}$$

[12] In the remainder, we consider the positive solution. Obviously, the derivation of the steady state entails non-negativity constraints on price and quantity, that we assume to be satisfied.

Now, substitute into (19) (i) $dp/dt = w\{\widehat{p}(t) - p(t)\}$, with $\widehat{p}(t) = A - NBq(t)$, where a symmetry assumption is introduced for individual firm's output; and (ii) $w\lambda_i(t) = [p(t) - c - q(t)]/B$ from (16). This yields:

$$\frac{dq(t)}{dt} = wA + (w+\rho)c - (2w+\rho)p(t) + [wB(1-N) + w + \rho]\,q(t) \tag{20}$$

Note that $dq(t)/dt = 0$ is a linear relationship between $p(t)$ and $q(t)$. This, together with $dp(t)/dt = 0$, also a linear function, fully characterise the steady state of the system. The dynamic system can be immediately rewritten in matrix form as follows:

$$\begin{bmatrix} \dot{p} \\ \dot{q} \end{bmatrix} = \begin{bmatrix} -w & -wBN \\ -(2w+\rho) & w+\rho-wB(N-1) \end{bmatrix} \begin{bmatrix} p \\ q \end{bmatrix} + \begin{bmatrix} wA \\ wA + (w+\rho)c \end{bmatrix} \tag{21}$$

As the determinant of the above 2×2 matrix is negative, the equilibrium point is a saddle, with

$$q^* = \frac{(A-c)(w+\rho)}{(w+\rho)(1+BN)+wB}\,;\; p^* = A - BNq^*\,. \tag{22}$$

4 Capital accumulation for production

Here, we present a model encompassing several contributions concerning the need for firms to invest in capital over time in order to produce the final goods to be supplied to consumers (Kamien and Schwartz, 1979; Fershtman and Muller, 1984; Cellini and Lambertini, 1998; see also Spence, 1979). The following exposition follows Cellini and Lambertini (1998).

Consider a market where N single-product firms offer differentiated products over $t \in [0, \infty)$. At any time t, the inverse demand function for variety i is (see Spence, 1976):

$$p_i(t) = A - Bq_i(t) - D\sum_{j \neq i} q_j(t) \tag{23}$$

where $D \in [0, B]$ is the symmetric degree of substitutability between any pair of varieties. If $D = B$, products are completely homogeneous; if $D = 0$, strategic interaction disappears and firms are independent monopolists. The direct demand function obtains by inverting (23):

$$q_i(t) = \frac{1}{B + D(N-1)} \cdot \left\{ A - \frac{[B + D(N-2)]p_i(t)}{B - D} + \frac{D}{B - D} \sum_{j \neq i} p_j(t) \right\}. \tag{24}$$

Producing any variety i requires physical capital k, accumulating over time to create capacity. At any t, the output level is $y_i(t) = f(k_i(t))$, with $f' \equiv \partial f(k_i(t))/\partial k_i(t) > 0$ and $f" \equiv \partial^2 f(k_i(t))/\partial k_i(t)^2 < 0$.

A reasonable assumption is that $q_i(t) \leq y_i(t)$, that is, the level of sales is at most equal to the quantity produced. Excess output is reintroduced into the production process yielding accumulation of capacity according to the following process:

$$\frac{\partial k_i(t)}{\partial t} = f(k_i(t)) - q_i(t) - \delta k_i(t), \tag{25}$$

where δ denotes the rate of depreciation of capital. In order to simplify further the analysis, suppose that unit variable cost is constant and equal to zero. The cost of capital is represented by the opportunity cost of intertemporal relocation of unsold output. Firm i's instantaneous profits i are

$$\pi_i(t) = p_i(t) q_i(t). \tag{26}$$

Firm i maximizes the discounted flow of its profits:

$$J_i = \int_0^\infty e^{-\rho t} \pi_i(t)\, dt \tag{27}$$

under the constraint (25) imposed by the dynamics of the state variable $k_i(t)$. Notice that the state variable does not enter directly the objective function. It can be assumed, alternatively, that all firms behave as either quantity-setters or price setters. Hence, the control variable is either $q_i(t)$ when all firms are Cournot agents, or $p_i(t)$ in the case where firms adopt a Bertrand behaviour.

4.1 Cournot competition

When firms compete in quantities, substitute (23) in (27) to get the relevant objective function of firm i:

$$J_i = \int_0^{\infty} e^{-\rho t}\, q_i(t) \cdot \left[A - Bq_i(t) - D \sum_{j \neq i} q_j(t) \right] \, dt \tag{28}$$

which must be maximised w.r.t. $q_i(t)$, under (25). The corresponding Hamiltonian function is:

$$\mathcal{H}(t) = e^{-\rho t} \cdot \left\{ q_i(t) \left[A - Bq_i(t) - D \sum_{j \neq i} q_j(t) \right] + \lambda_i(t) \left[f(k_i(t)) - q_i(t) - \delta k_i(t) \right] \right\}, \tag{29}$$

where $\lambda_i(t) = \mu_i(t) e^{\rho t}$, and $\mu_i(t)$ is the co-state variable associated to $k_i(t)$.

The solution of firm i's problem follows from the above conditions (6), (7) and (8), appropriately written for the present model. Specifically, the necessary and sufficient conditions for a path to be optimal are:

$$\frac{\partial \mathcal{H}(t)}{\partial q_i(t)} = A - 2Bq_i(t) - D \sum_{j \neq i} q_j(t) - \lambda_i(t) = 0; \tag{30}$$

$$-\frac{\partial \mathcal{H}(t)}{\partial k_i(t)} = \frac{\partial \mu_i(t)}{\partial t} \Rightarrow \frac{\partial \lambda_i(t)}{\partial t} = \left[\rho + \delta - f'(k_i(t)) \right] \lambda_i(t); \tag{31}$$

$$\lim_{t \to \infty} \mu_i(t) \cdot k_i(t) = 0\,. \tag{32}$$

From (30) we obtain

$$q_i(t) = \frac{A - D \sum_{j \neq i} q_j(t) - \lambda(t)}{2B} \tag{33}$$

which can be differentiated w.r.t. time to get

$$\frac{dq_i(t)}{dt} = \frac{-D \sum_{j \neq i} (dq_j(t)/dt) - d\lambda_i(t)/dt}{2B}. \tag{34}$$

Thanks to (31), the expression in (34) simplifies as follows:

$$\frac{dq_i(t)}{dt} = \frac{1}{2B} \left[\left(f'(k_i(t)) - \rho - \delta \right) \lambda_i(t) - D \sum_{j \neq i} \frac{dq_j(t)}{dt} \right]. \tag{35}$$

In order to simplify calculations and to obtain an analytical solution, we adopt the following assumption, based on firms' *ex ante* symmetry:

$$\sum_{j \neq i} q_j(t) = (N-1)q_i(t) \tag{36}$$

so that

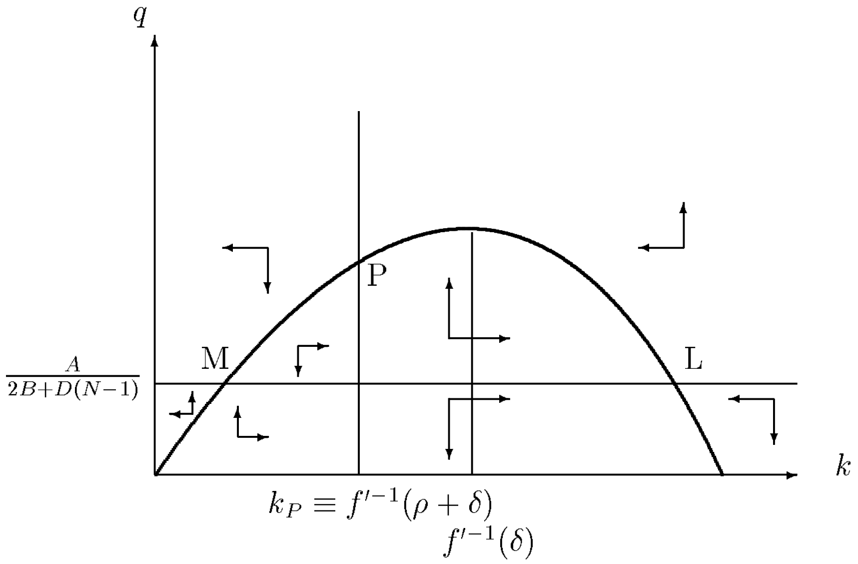
$$\sum_{j \neq i} \frac{dq_j(t)}{dt} = \frac{(N-1)dq_i(t)}{dt}.$$

Thanks to symmetry, in the remainder we drop the indication the identity of the firm. As a further simplification, we also drop the indication of time. Using (36) and (31), we rewrite (35) as follows:

$$\frac{dq}{dt} = (f'(k) - \rho - \delta) \cdot \frac{A - (2B + D(N-1))q}{2B + D(N-1)}. \tag{37}$$

We are now able to draw a phase diagram in the space $\{k, q\}$, in order to characterise the steady state equilibrium. The locus $\dot{q} \equiv dq/dt = 0$ is given by $q = A/(2B + D(N-1))$ and $f'(k) = \rho + \delta$ in figure 1. Notice that the horizontal locus $q = A/(2B + D(N-1))$ denotes the usual equilibrium solution we are well accustomed with from the existing literature dealing with static market games (see, e.g., Singh and Vives, 1984; Majerus, 1988). The two loci partition the space $\{k, q\}$ into four regions, where the dynamics of q is determined by (37), as summarised by the vertical arrows. The locus $\dot{k} \equiv dk/dt = 0$ as well as the dynamics of k, depicted by horizontal arrows, derive from (25). Steady states, denoted by M, L along the horizontal arm, and P along the vertical one, are identified by intersections between loci.

Figure 1: Cournot competition



It is worth noting that the situation illustrated in figure 1 is only one out of five possible configurations, due to the fact that the position of the vertical line $f'(k) = \rho + \delta$ is independent of demand parameters, while the horizontal locus $q = A/(2B + D(N-1))$ shifts upwards (downwards) as A (B, D and N) increases. Therefore, we obtain one out of five possible regimes:

[1]. There exist three steady state points, with $k_M < k_P < k_L$ (this is the situation depicted in figure 1).

[2]. There exist two steady state points, with $k_M = k_P < k_L$.

[3]. There exist three steady state points, with $k_P < k_M < k_L$.

[4]. There exist two steady state points, with $k_P < k_M = k_L$.

[5]. There exists a unique steady state point, corresponding to P.

An intuitive explanation for the above taxonomy can be provided, in the following terms. The vertical locus $f'(k) = \rho + \delta$ identifies a constraint on optimal capital embodying firms' intertemporal preferences, i.e., their common discount rate. Accordingly, maximum output level in steady state would be that corresponding to (i) $\rho = 0$, and (ii) a capacity such that $f'(k) = \delta$. Yet, a positive discounting (that is, impatience) induces producers to install a smaller steady state capacity, much the same as it happens in the well known Ramsey model (Ramsey, 1928).[13] On these grounds, define this level of k as the *optimal capital constraint*, and label it as $\widehat{k}$. When the reservation price A is very large (or B, D, N, are low), points M and L either do not exist (regime [5]) or fall to the right of P (regimes [2], [3], and [4]). Under these circumstances, the capital constraint is operative and firms choose the capital accumulation corresponding to P. As we will see below, this is fully consistent with the dynamic properties of the steady state points.

Notice that, since both steady state points located along the horizontal locus entail the same levels of sales. As a consequence, point L is surely inefficient in that it requires a higher amount of capital. Point M, as already mentioned above, corresponds to the optimal quantity emerging from the static version of the game. It is hardly the case of emphasising that this solution encompasses both monopoly (either when $N = 1$ or when $D = 0$) and perfect competition (as, in the limit, $N \to \infty$).[14] In point M, $d\pi_i(t)/dq_i(t) = 0$, that is, marginal instantaneous profit is nil.

Now we come to the stability analysis of the above system. The joint dynamics of q and k, can be described by linearising (37) and (25) around (k^*, q^*), to get what follows:

$$\begin{bmatrix} \dot{k} \\ \dot{q} \end{bmatrix} = \Xi \begin{bmatrix} (k - k^*) \\ (q - q^*) \end{bmatrix} \tag{38}$$

where

$$\Xi = \begin{bmatrix} f'(k) - \delta & -1 \\ \frac{A-(2B+D(n-1))q}{2B+D(n-1)} f''(k) & -(f'(k) - \rho - \delta) \end{bmatrix}$$

[13]For a detailed exposition of the Ramsey model, we refer the reader to Blanchard and Fischer (1989, ch. 2).

[14]The analysis of dynamic monopoly with capital accumulation dates back to Evans (1924). See Chiang (1992) for a recent exposition of the original model by Evans, as well as later developments.

The stability properties of the system in the neighbourhood of the steady state depend upon the trace and determinant of the 2×2 matrix Ξ. In studying the system, we confine to steady state points. The trace of Ξ is $tr(\Xi) = \rho > 0$, whereas the determinant $\Delta(\Xi)$ varies according to the point where it is evaluated. Consider the above taxonomy.

Regime [1]. In M, $\Delta(\Omega) < 0$, hence this is a saddle point. In P, $\Delta(\Omega) > 0$, so that P is an unstable focus. In L, $\Delta(\Omega) < 0$, and this is again a saddle point, with the horizontal line as the stable arm.

Regime [2]. In this regime, M Coincides with P, so that we have only two steady states which are both are saddle points. In $M = P$, the saddle path approaches the saddle point from the left only, while in L the stable arm is again the horizontal line.

Regime [3]. Here, P is a saddle; M is an unstable focus; L is a saddle point, as in regimes [1] and [2].

Regime [4]. Here, points M and L coincide. P remains a saddle, while $M = L$ is a saddle whose converging arm proceeds from the right along the horizontal line.

Regime [5]. Here, there exists a unique steady state point, P, which is also a saddle point.

We can sum up the above discussion as follows. The unique efficient and non-unstable steady state point is P if $k_P < k_M$, while it is M if the opposite inequality holds. Such a point is always a saddle. Individual equilibrium output is $q_M^C = A/(2B+D(N-1))$ if the equilibrium is identified by point M, or the level corresponding to the optimal capital constraint $\widehat{k}$ if the equilibrium is identified by point P. The reason is that, if the capacity at which marginal instantaneous profit is nil is larger than the optimal capital constraint, the latter becomes binding. Otherwise, the capital constraint is irrelevant, and firms' decisions in each period are solely driven by the unconstrained maximisation of single-period profits. It is apparent that, in the present setting, firms always operate at full capacity.[15] When optimal output is q_M^C, per-firm

[15]The possibility for firms to choose capacity strategically has been extensively debated in static models (Levitan and Shubik, 1972; Kreps and Scheinkman, 1983; Davidson and Deneckere, 1986; Osborne and Pitchick, 1986).

instantaneous profits in steady state are

$$\pi_M^C = \frac{A^2 B}{[2B + D(N-1)]^2} \tag{39}$$

while they are $\pi_P^C = \widehat{k}\left\{A - [B + D(N-1)]\,\widehat{k}\right\}$ if optimal output is $\widehat{k}$.[16]

4.2 Bertrand competition

Consider now the alternative setting where firms compete in prices. In this case, the demand function is (24), and firm i's dynamic problem is:

$$\max_{p_i(t)} \quad J_i = \int_0^\infty \frac{e^{-\rho t}\, p_i(t)}{B + D(N-1)} \cdot \left\{A - \frac{[B + D(N-2)]p_i(t)}{B - D} + \frac{D}{B - D}\sum_{j \neq i} p_j(t)\right\} dt \tag{40}$$

$$s.t. \quad \dot{k_i}(t) = f(k_i(t)) - \delta k_i(t) +$$

$$-\left\{\frac{A}{B + D(N-1)} - \frac{[B + D(N-2)]p_i(t)}{[B + D(N-1)](B - D)} + \frac{D}{[B + D(N-1)](B - D)}\sum_{j \neq i} p_j(t)\right\}. \tag{41}$$

The corresponding Hamiltonian function is now relatively straightforward. On the basis of FOCs, and using the symmetry assumption $\sum_{j \neq i} p_j = (N-1)p_i$, the necessary and sufficient conditions for the optimal path obtain:

$$\dot{p} \equiv \frac{dp}{dt} = 0; \ \dot{k} \equiv \frac{dk}{dt} = 0, \tag{42}$$

along with the standard transversality condition

$$\lim_{t \to \infty} \vartheta(t) \cdot k(t) = 0, \tag{43}$$

where $\vartheta(t)$ is the co-state variable associated with $k(t)$. The explicit derivation of expressions (42) is left to the reader, as well as the pertaining phase

[16] Our approach generalises the analysis in Fershtman and Muller (1984). They do not consider demand conditions, and suppose that instantaneous individual profits are everywhere increasing in each firm's own capital (see their assumption 2, p. 325). In our terminology, they only identify the equilibrium in P along the vertical locus $f'(k) = \rho + \delta$. They are prevented from reaching an equilibrium like point M, in that the horizontal locus $q = A/(2B + D(N-1))$ does not appear in their model.

diagram in the space $\{k, p\}$. The locus $\dot{k}=0$ is a convex curve, while the locus $\dot{p}=0$ consists of the two orthogonal lines along which $f'(k) = \rho + \delta$ (as in the Cournot case) and

$$p^* = \frac{A(B-D)}{2(B-D)+D(N-1)}, \tag{44}$$

respectively. The analysis of the price-setting case is qualitatively analogous to the case of quantity-setting behaviour. There exist one, two or three steady state points, according to the relative position of the two loci. From the analysis of the dynamic properties of the system, we draw the following conclusions. The unique efficient and non-unstable steady state point is P if $k_P < k_M$, while it is M if the opposite holds. This is always a saddle point. Individual equilibrium output is

$$q_M^B = \frac{A\left[B+D(N-2)\right]}{\left(2(B-D)+D(N-1)\right)\left(B+D(N-1)\right)} \tag{45}$$

in M, or the level corresponding to the optimal capacity constraint $\widehat{k}$ in P. In the former case, instantaneous steady state profits per firm are

$$\pi_M^B = \frac{A^2(B-D)\left[B+D(N-2)\right]}{\left[2(B-D)+D(N-1)\right]^2\left[B+D(N-1)\right]} \tag{46}$$

while $\pi_P^B = \pi_P^C = \widehat{k}\left\{A - \left[B+D(N-1)\right]\widehat{k}\right\}$ if optimal output is $\widehat{k}$.

4.3 The social optimum

From a social planner's viewpoint, the choice between prices and quantities is completely immaterial. Moreover, in this case the symmetry assumption can be adopted from the outset, so that market demand for each product writes as $p = A - (B + D(N-1))q$, where q is the individual firm's level of sales. Instantaneous social welfare, defined as the sum of consumer surplus and firms' profits, is

$$sw(t) = \frac{Nq(t)}{2}\left[2A - (B+D(N-1))q(t)\right]. \tag{47}$$

The resulting optimum problem for the social planner can be written as follows:

$$\max_q \quad SW = \int_0^\infty e^{-\rho t} sw(t)\, dt \tag{48}$$

$$s.t. \quad \frac{\partial k(t)}{\partial t} = f(k(t)) - q(t) - \delta k(t). \tag{49}$$

The solution of the social optimum problem is formally equivalent to what we carried out in the section dealing with Cournot behaviour under all respects, with the exception of the unconstrained optimal sales level, which here is

$$q_M^S = \frac{A}{B + D(N-1)} \tag{50}$$

This output level is obviously larger than both the Cournot and the Bertrand levels, if the capital constraint is not binding, while the three regimes are indistinguishable from one another when the capital constraint becomes operative in the Cournot setting. Intuitively, there can be cases where the constraint binds under social planning and or Bertrand behaviour but not under Cournot competition. Hence, if $k_P > k_M$ in all regimes, then $q_M^S > q_M^B > q_M^C$. Iff $k_P \leq k_M$ in the Cournot setting, then the optimal efficient point corresponds to $k_P = f'^{-1}(\rho + \delta)$ in all regimes, with $q_P^S = q_P^B = q_P^C$.

A straightforward implication of the above proposition is that, when the capital accumulation constraint comes into operation in all settings, the three regimes are observationally equivalent. In particular, the following relevant conclusion can be drawn. If $k_P > k_M$ in all regimes, then $SW^S \geq SW^B \geq SW^C$; otherwise, $SW^S = SW^B = SW^C$.[17] The first chain of inequalities on the ranking of social welfare levels across regimes replicates the established wisdom according to which social planning is more efficient than Bertrand competition, and both are more efficient than Cournot competition, as long as products are differentiated and the number of firms is finite. In this case, social welfare is the same irrespectively of the market regime, only when the number of firms becomes infinitely large. Social planning and Bertrand competition coincide also when products are perfect substitutes. The second result in the above corollary indicates that market conditions are irrelevant if the allocation of resources is driven only by the dynamic accumulation constraint.

[17] As it is usually done in the existing literature on static oligopoly competition, these inequalities are calculated *for a given number of firms* (see Vives, 1985; Okuguchi, 1987). It can be shown that the incentive to enter vanishes faster under price than under quantity competition, so that we might expect the number of firms to be larger in the Cournot steady state. This may reverse the above inequalities on output and social welfare levels, and make Cournot socially more desirable than Bertrand (see Cellini, Lambertini and Ottaviano, 1999).

5 Product and process innovation

Research in the economics of innovation has focused upon two different issues, process and product innovation, the first having received more attention than the second. However, the differential game approach to both problems has produced relatively few contributions. In this section, we present two models, dealing, respectively, with (i) product innovation in a framework of perfect certainty and (ii) a stochastic race for a generic technological breakthrough, that might turn into either a new product or a new (and cheaper) production process for existing products.

5.1 R&D activity for product innovation

Here, we use the same demand structure as in section 4, except that we assume that the degree of substitutability, D, is the result of R&D activity. Thus, D is the steady state variable common to all firms, and we suppose that there is no capacity constraint on firms' output. Notice that investing to reduce D amounts to investing in product differentiation. We investigate two alternative situations. In the first, firms take their decisions noncooperatively, with respect to both the R&D investment and the market behaviour. Here, we reach an Arrowian conclusion according to which the amount of resources invested by the industry in product differentiation is increasing in the number of firms, i.e., in the intensity of market competition. In the second setting, we model the behaviour of an R&D cartel made up by all firms, which continue to behave noncooperatively in setting their respective output levels. In this case, the main result is that the R&D cartel invests more than the sum of independent ventures, and therefore obtains a higher degree of product differentiation.

5.1.1 The setup

We use the same setup as in the previous section. Consider a market where N single-product firms sell differentiated products over $t \in [0, \infty)$. Market competition takes place *à la* Cournot. The demand structure is (23). At any time t, the output level $q_i(t)$ is produced at constant returns to scale, for a given D, and we normalise marginal (and average) cost to zero.

We assume that, at the initial instant $t = 0$, $D = B$, so that firms may produce the same homogeneous good through a technology which is public domain.[18] Product differentiation may increase (that is, D may decrease) through firms' R&D investment according to:

$$\frac{\partial D(t)}{\partial t} = -\frac{K(t)}{1+K(t)} \cdot D(t) \equiv -\frac{k_i(t) + \sum_{j \neq i} k_j(t)}{1 + \left[k_i(t) + \sum_{j \neq i} k_j(t)\right]} \cdot D(t); \quad k_i(t) \geq 0 \; \forall \; i. \tag{51}$$

The above dynamics of product differentiation can be interpreted as follows. The industry overall R&D expenditure is $K(t)$, while $k_i(t)$ is individual investment. Given the symmetric nature of product differentiation in this model, there exists a complete spillover effect in the R&D process. Notice that the externality effect we consider here entails that the outcome of R&D activity is public domain via the demand function. On the contrary, the externality effects usually considered in the literature are associated with information leakage or transmission (see, *inter alia*, d'Aspremont and Jacquemin, 1988). The R&D technology defined by (51) exhibits decreasing returns to scale. As a result, $D(t)$ is non-increasing over time, and would approach zero if $K(t)$ tended to infinity.

The instantaneous profit is $\pi_i(t) = p_i(t)q_i(t) - k_i(t)$. Each firm aims at maximizing the discounted value of its flow of profits $J_i = \int_0^\infty e^{-\rho t}\pi_i(t)\, dt$ under the dynamic constraint (51) concerning the state variable $D(t)$. The control variables are $q_i(t)$ and $k_i(t)$.

5.1.2 Non-cooperative R&D

Suppose firms choose non-cooperatively both R&D efforts and output levels. The solution concept we adopt is the open-loop Nash equilibrium. The objective function of firm i is:

$$J_i = \int_0^\infty e^{-\rho t} \left\{ q_i(t) \cdot \left[A - Bq_i(t) - D(t) \sum_{j \neq i} q_j(t) \right] - k_i(t) \right\} dt \tag{52}$$

to be maximised w.r.t. $q_i(t)$ and $k_i(t)$, under (51). The corresponding Hamiltonian function is:

$$\mathcal{H}(t) = e^{-\rho t} \cdot \{ Aq_i(t) - B(q_i(t))^2 - D(t)q_i(t) \sum_{j \neq i} q_j(t) - k_i(t) + \tag{53}$$

[18] The idea that D depends upon the behaviour of firms has been investigated in static models by Harrington (1995); Lambertini and Rossini (1998); Lambertini, Poddar and Sasaki (1998).

$$+\lambda_i(t)[-\frac{k_i(t)+\sum_{j\neq i}k_j(t)}{1+\left[k_i(t)+\sum_{j\neq i}k_j(t)\right]}\cdot D(t)]\},$$

where $\lambda_i(t) = \mu_i(t)e^{\rho t}$, $\mu_i(t)$ being the co-state variable associated to $D(t)$. Necessary and sufficient conditions for a path to be optimal are:

$$\frac{\partial \mathcal{H}(t)}{\partial q_i(t)} = A - 2Bq_i(t) - D(t)\sum_{j\neq i} q_j(t) = 0; \tag{54}$$

$$\frac{\partial \mathcal{H}(t)}{\partial k_i(t)} = -1 - D(t)\lambda_i(t)\frac{1}{\left(1+k_i(t)+\sum_{j\neq i}k_j(t)\right)^2} = 0; \tag{55}$$

$$-\frac{\partial \mathcal{H}(t)}{\partial D(t)} = \frac{\partial \mu_i(t)}{\partial t} \Rightarrow \frac{\partial \lambda_i(t)}{\partial t} = q_i(t)\sum_{j\neq i} q_j(t) + \lambda_i(t)\left(\frac{k_i(t)+\sum_{j\neq i}k_j(t)}{1+\left[k_i(t)+\sum_{j\neq i}k_j(t)\right]} + \rho\right); \tag{56}$$

$$\lim_{t\to\infty} \mu_i(t)\cdot D(t) = 0 \tag{57}$$

We introduce the usual symmetry assumption involving no loss of generality: $q_i(t) = q_j(t) = q(t)$, and $k_i(t) = k_j(t) = k(t)$. This implies $\sum_{j\neq i} q_j(t) = (N-1)q(t)$ and $\sum_{j\neq i} k_j(t) = (N-1)k(t)$.

From (54) we get the individual equilibrium output:[19]

$$q(t) = \frac{A}{2B + (N-1)D(t)} \tag{58}$$

Hence, (55) rewrites as

$$-\lambda(t) = \frac{\left[1+Nk(t)\right]^2}{D(t)}. \tag{59}$$

Likewise, (56) simplifies as follows:

$$\frac{\partial \lambda(t)}{\partial t} = (N-1)\left[q(t)\right]^2 + \frac{N\lambda(t)k(t)}{1+Nk(t)} + \lambda(t)\rho. \tag{60}$$

[19] Which, again, coincides with the standard outcome of Cournot models with product differentiation (Singh and Vives, 1984; Majerus, 1988).

From (59) we obtain $k(t)$, which can be differentiated w.r.t. t. Then, plugging (60) into $\partial k(t)/\partial t$, one obtains:

$$\frac{\partial k(t)}{\partial t} = \frac{1}{2N}\sqrt{\frac{D(t)}{-\lambda(t)}} \cdot \left\{-\lambda(t)\rho - (N-1)[q(t)]^2\right\}. \tag{61}$$

This can be further simplified by substituting the co-state variable with (59), to get:

$$\frac{\partial k(t)}{\partial t} = \frac{1}{2n(1+Nk(t))} \cdot \left\{\frac{\rho}{D(t)}[1+Nk(t)]^2 - (N-1)[q(t)]^2\right\}. \tag{62}$$

which obviously holds for all $D(t) \in (0, B]$. If $D(t) = 0$, optimal per-period investment is $k(t) = 0$.

We are now in a position to assess the overall dynamic properties of the model, fully characterised by (62) and $\partial D(t)/\partial t = -Nk(t)D(t)/(1+Nk(t))$. The latter equation establishes that $\partial D(t)/\partial t < 0$ for all $k(t) \in (0, \infty)$ and for all $D(t) \in (0, B]$; while $\partial D(t)/\partial t = 0$ if $k(t) = 0$ or $D(t) = 0$. In the latter case, it is immediate to verify that $\partial q(t)/\partial t$ is also nil. Moreover,

$$\text{sign}\left\{\frac{\partial k(t)}{\partial t}\right\} = \text{sign}\left\{\frac{\rho}{D(t)}[1+Nk(t)]^2 - (N-1)[q(t)]^2\right\}. \tag{63}$$

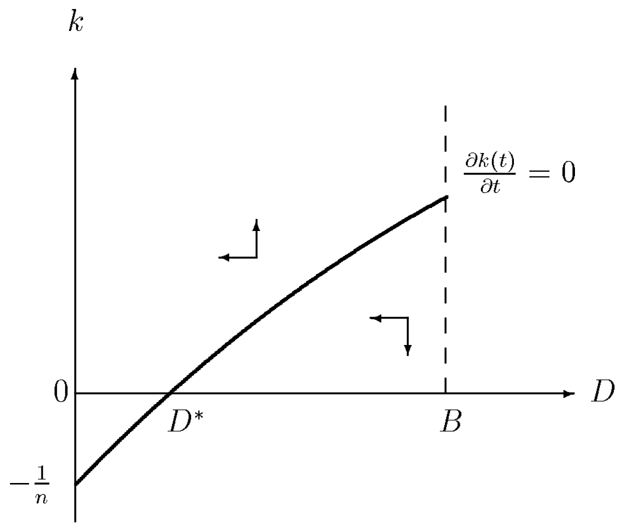
Thus, using equilibrium output (58), we have:

$$\frac{\partial k(t)}{\partial t} > 0 \text{ iff } k(t) > \frac{1}{N}\left[\frac{A\sqrt{(N-1)D(t)}}{[2B+(N-1)D(t)]\sqrt{\rho}} - 1\right]. \tag{64}$$

We are interested in investigating the dynamics of the system in the positive quadrant of the space $\{D, k\}$, which is described in figure 2. The locus $\partial D(t)/\partial t = 0$ corresponds to the axes. The locus $\partial k(t)/\partial t = 0$ draws a curve over the admissible range of parameter D, which may or may not cross the horizontal axis within the same range, i.e., $D \in (0, B]$. If it does, the resulting degree of substitutability in steady state is

$$D^* = \frac{A^2 - 4B\rho - (A-c)\sqrt{(A-c)^2 - 8B\rho}}{2(N-1)\rho}. \tag{65}$$

Figure 2 : Dynamics in the space (D, k)



See Cellini and Lambertini (1999) for details concerning the conditions ensuring that $D^* \in (0, B]$. When no steady state exists, the model becomes trivial, in that the only admissible strategy is $k(t) = 0$ at every t, implying that firms are stuck with homogeneous products forever.

As to the stability of the system, it remains to be stressed that, whenever $D^* \in (0, B]$, it is a saddle, and it can obviously be approached only along the north-east arm of the saddle path.

We now proceed to the comparative statics on D^* w.r.t. all parameters.

From (65), it is immediately verified that, *ceteris paribus*, D^* is decreasing, i.e., steady state product differentiation is increasing, in the number of firms. This result can be interpreted in the light of the debate between the polar positions of Schumpeter (1942) and Arrow (1962), concerning the relationship between the intensity of market competition and the incentives to invest in R&D (for a survey, see Reinganum, 1989). Here, R&D efforts are aimed at increasing product differentiation. In general, any increase in the number of firms lowers profits, and this tendency can be counterbalanced by investing a larger amount of resources in order to decrease the degree of substitutability among products. Notice that the anti-Schumpeterian flavour of these considerations is evident, in the limit case $N = 1$, when the monopolist has no incentive at all to invest.

Not surprisingly, $\partial D^*/\partial A > 0$ and $\partial D^*/\partial B < 0$. Given that A/B yields a measure of market size and profitability, any increase in this ratio induces firms to reduce their expenditure in product differentiation.

Finally, $\partial D^*/\partial \rho < 0$ can be interpreted in the following terms. As ρ becomes higher, the present value of future profits shrinks. This can be balanced by a higher investment in product differentiation. More explicitly, an increase in ρ seemingly reduces, *ceteris paribus*, firms capability to spend as measured by the incoming profit flows. However, a reduction in D^* does indeed restore endogenously firms' profitability and, consequently, their incentive to invest so as to offset the negative effects produced by higher discounting.

5.1.3 Cooperative R&D

Here, we assume that firms behave noncooperatively in the marketing phase, while they activate a cartel in the R&D investment phase. As in previous literature (d'Aspremont and Jacquemin, 1988; Kamien, Muller and Zang, 1992), we assume the cartel to coordinate the R&D expenditure of all firms so as to maximise industry profits in the R&D phase.

This entails maximising

$$J_i = \int_0^\infty e^{-\rho t} \left\{ q_i(t) \cdot \left[A - Bq_i(t) - D(t) \sum_{j \neq i} q_j(t) \right] - k(t) \right\} dt \tag{66}$$

subject to the constraint:

$$\frac{\partial D(t)}{\partial t} = -\frac{K(t)}{1+K(t)} \cdot D(t) \equiv -\frac{Nk(t)}{1+Nk(t)} \cdot D(t); \quad k(t) \geq 0\,. \tag{67}$$

Notice that $k(t)$ has no subscript in that we impose symmetry across firms in the R&D phase from the outset, in order to capture the idea that the R&D cartel optimises w.r.t. $K(t)$ and then symmetrically charges each firm of her share $k(t) = K(t)/n$ of the overall expenditure. It is also worth stressing that this procedure is equivalent to what we usually observe in static R&D models (Katz, 1986; d'Aspremont and Jacquemin, 1988; Kamien, Muller and Zang, 1992; Suzumura, 1992, *inter alia*), where firms' behaviour is described by a two-stage game. In the first, a single agent (the cartel) chooses the symmetric investment level maximising cartel profits; in the second, firms compete in the relevant market variable.

Adopting the usual procedure, we find that the candidate steady state level of D is

$$\widehat{D} = \frac{N(A-c)^2 - 4B\rho - A\sqrt{N^2(A-c)^2 - 8B\rho N}}{2(N-1)\rho}. \tag{68}$$

Then, $\widehat{D} \in [0, B]$ iff $A^2 \geq (N+1)^2 B\rho/[N(N-1)]$. Again, see Cellini and Lambertini (1999) for further details.

We are now able to characterise the main result, namely, that $\widehat{D} < D^*$ when both exist. This amounts to saying that product substitutability in steady state is lower under R&D cooperation than under noncooperative behaviour. Accordingly, both the per-firm and the aggregate steady state output level is larger when an R&D cartel operates.

This has some relevant implications as to the established wisdom on the investment behaviour of R&D cartels, according to which an R&D cartel invests less than a decentralised industry, if technological spillovers are low enough, and conversely.[20] Hence, with low spillovers, an R&D cartel can be successful in mitigating the well known wasteful duplication of efforts affecting competitive industries. However, this literature deals with process rather than product innovations. Consequently, spillover effects operate within the R&D technology. In our setting, there exists a full spillover effect from each firm's investment to her market mates through consumer preferences, i.e., there is no wasteful duplication of efforts.[21] Therefore, the ultimate implication of the externality is to drive the R&D cartel's investment well beyond that resulting from independent ventures. An alternative viewpoint to interpret the above results is that, given the noncooperative behaviour of firms in setting the output levels, cartelisation in the development phase can produce higher profits for its members only by increasing differentiation.

[20] Cf. d'Aspremont and Jacquemin (1988, p. 1135); Kamien, Muller and Zang (1992, proposition 1, p. 1301). See also Katz and Ordover (1990), for an exhaustive survey.

[21] Setting up a research joint venture would indeed eliminate duplication completely, in that firms would jointly proceed to the development of a single good. In the present model this would trivially imply that they should not invest at all and market the undifferentiated good available at $t = 0$.

5.2 R&D races under uncertainty

So far, we have treated innovation as an enterprise whose outcome is perfectly known from the outset. However, one could stress that innovation is an uncertain adventure. Firms are subject to technological uncertainty, in that they are unable to foresee with certainty the R&D investment globally required for any of them to achieve an innovation. In addition to this, each firm is subject to the uncertainty associated with the possibility of a rival's pre-emptive breakthrough. Accordingly, R&D activity has been modelled in stochastic environments in the differential game approach. Kamien and Schwartz (1972; 1976) and Reinganum (1981; 1982a) are the most relevant contributions in the field of stochastic differential games of innovation. Here, we broadly follow Reinganum (1981, 1982a). Suppose N firms compete over $t \in [0, T]$ for a technological innovation, that might lead either to a new and cheaper production process for an existing commodity, or to a new good. Suppose the innovation date for firm i is a random variable τ_i distributed according to $F_i(t) = \Pr\{\tau_i \leq t\}$, with dates τ_i being i.i.d.. The model is worked out under the assumption that the *innovator* gets a patent of infinite duration over the innovation, but there exists the possibility of imitation due to imperfect patent protection, so that all other firms may continue to operate with the technology already available before the innovation, or imitate the innovator.[22] If the innovation occurs at $\tau = \min_i \tau_i$, the innovator is firm j with $\tau_j = \tau$. Independency implies:

$$F(t) = \Pr\{\tau \leq t\} = 1 - \Pi_{i=1}^{N} [1 - F_i(t)] \tag{69}$$

We define as $k_i(t)$ the intensity of the research effort of firm i at time t, with the R&D cost being $C_i(t) = [k_i(t)]^2 /2$, and introduce the assumption that firm i's conditional probability of innovation at date t (the *hazard rate*) is

$$\frac{dF_i(t)}{dt} \equiv \dot{F}_i (t) = \beta k_i(t) [1 - F_i(t)] \; ; \; \beta > 0; \; F_i(0) = 0 \, . \tag{70}$$

In contrast with the models reviewed so far, here the market interaction and the resulting per-period profits are blackboxed. Define the present value

[22] This amounts to saying that the profits from the innovation accrues solely to the winner, but the innovation is *non-drastic*, i.e., it does not create a monopoly for the innovator. In this respect, we follow Reinganum (1982a). For the consequences of the alternative assumption that *the winner takes all*, see Reinganum (1981). For further discussion on these issues, see Tirole (1988, ch. 10) and Reinganum (1989), *inter alia*.

of the innovation to the winner as V_W, and the present value to the loser(s) of an alternative technology as V_L.[23] The expected profit flow of firm i is then:

$$J_i^e = \int_0^T \{V_W \dot{F}_i(t) \prod_{j\neq i} [1 - F_i(t)] + V_L \sum_{j\neq i} \dot{F}_j(t) \prod_{m\neq j} [1 - F_j(t)] + \tag{71}$$

$$-\frac{[k_i(t)]^2}{2e^{\rho t}} \prod_{j=1}^{N} [1 - F_j(t)]\}dt$$

In order to simplify the exposition, define $\ln[1 - F_i(t)] = -\beta x_i(t)$, so that it is possible to write firm i's problem as:[24]

$$J_i^e = \int_0^T \exp\left\{-\beta \sum_{j=1}^{N} x_j(t)\right\} \times \left[\beta V_W k_i(t) + \beta V_L \sum_{j\neq i} k_j(t) - \frac{[k_i(t)]^2}{2e^{\rho t}}\right] dt \tag{72}$$

$$s.t.\ \dot{x} \equiv \frac{dx_i(t)}{dt} = k_i(t)\,,\ x_i(0) = 0 \tag{73}$$

The game characterised as in (72-73) has a unique open loop Nash equilibrium. Using the following transformation:

$$\eta = \exp\left\{-\alpha\beta \sum_{j=1}^{N} x_j(t)\right\} \tag{74}$$

and provided $\alpha \neq 0$, we may write the Hamiltonian function:

$$\mathcal{H}_i(t) = \eta^{1/\alpha} \cdot \left[\beta V_W k_i(t) + \beta V_L \sum_{j\neq i} k_j(t) - \frac{[k_i(t)]^2}{2e^{\rho t}}\right] - \mu_i \alpha\beta\eta - \sum_{j=1}^{N} k_j(j) \tag{75}$$

from which we obtain:

$$k_i^*(t, \alpha, \eta) = \left[V_W - \alpha\lambda_i(t)\eta^{(\alpha-1)/\alpha}\right] \beta e^{\rho t} \tag{76}$$

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[23] The payoff V_L could be the profit resulting from the use of an inferior technology available after the innovator's breakthrough. Obviously, $V_L < V_W$.

[24] Reinganum (1981, 1982a) assumes that, for any firm i, the probability of being the innovator at any date t is positively related to the amount of knowledge $\kappa_i(t)$ accumulated by the same date:

$$\Phi_i[\kappa_i(t)] = 1 - \exp\{-\beta\kappa_i(t)\}$$

with $\dot{\kappa} \equiv d\kappa_i(t)/dt = \nu_i(t)$ being the rate at which kwoledge accumulates over time.

where $\lambda_i(t)$ must satisfy:

$$\dot{\lambda}_i \equiv \frac{d\lambda_i(t)}{dt} = -\frac{\partial \mathcal{H}_i}{\partial \eta} - \sum_{j \neq i} \frac{\partial \mathcal{H}_i}{\partial k_i} \cdot \frac{\partial k_i^*}{\partial \eta} \; ; \; \lambda_i(T) = 0 \tag{77}$$

From (76) it can be ascertained that increasing the prize (V_W) for the innovation induces firm i to increase the (optimal) R&D effort. The opposite obviously holds for an increase in α and/or η.

A cooperative solution can also be adopted (see Reinganum, 1981, pp. 31-33). In such a case, firms would maximise joint profits w.r.t. the collective R&D effort. Reinganum (1981, pp. 34-36) shows that R&D cooperation allows firms to reduce wasteful effort duplication. Put it the other way around, rivalry induces player to invest in R&D at a uniformly higher rate than cooperation does. The other side of the hill is that noncooperative behaviour allows firms to innovate earlier than they would under cooperative behaviour.[25] Hence, the question whether cooperation is better than rivalry is elusive, and so are the implications for antitrust policy and R&D subsidisation as well.

6 Concluding remarks

In this chapter, we have provided the reader with a summary of the toolbox of differential game theory, with a brief collection of examples of its applications to oligopoly settings. Although exhaustiveness is far beyond the scope of our exposition, we believe that the foregoing overview suffices to grasp the investigative power of differential game theory with respect to the research currently undertaken in the field of industrial organization. In particular, differential games properly highlight the role of time in strategic interactions where some form of capital accumulates over time. This feature remains often out of reach in static multi-stage models, where, by definition, no costly dynamic accumulation exists.

[25] This holds when the outcomes of innovative activity can be privately retained. The conclusion is completely reversed if the results obtained by R&D activity are public domain, i.e., if they fully leak out to rivals.

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