Monopoly
à la Hotelling

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Abstract

In this short note, the behaviour of a monopolist in a horizontally differentiated market is analysed. It turns out that her incentive to differentiate is nihil, while both the optimal location choice and the optimal pricing rule depend on the relative size of the parameters involved in the model.

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1. Introduction

The literature takling the issue of horizontal differentiation stems from Hotelling's duopoly (1929), in which the hypothesis of linear tranportation costs led to the so called *minimum differentiation principle*; the Bertrand paradox implicit in this result was given a solution by assuming quadratic transportation costs (D'Aspremont, Gabszewicz and Thisse, 1979). The aim of this short note is to investigate the behaviour of a monopolist in a horizontally differentiated market, i.e., to identify both the optimal location choice and the optimal pricing rule of a monopolist whose objective function is the sum of the duopolists', in the standard quadratic transportation cost framework.

2. The Model

The monopolist has two stores, which sell the same physical good. Assume consumers be uniformly distributed along an interval of lenght 1 (a 'linear city', or Hotelling's beach), with total density equal to 1. Consumers have unit demands, and consumption yields a positive constant gross surplus s; then each consumer buys if and only if the following condition is satisfied:

$$U = s - tx^2 - p_i \geq 0, \quad 0 \leq x \leq 1, \quad t > 0, \quad i = 1,2; \tag{1}$$

i.e., if the net utility derived from consumption is non-negative; tx^2 is the transportation cost incurred by a consumer living at distance x from store i (the closest one); p_i is the 'mill price' charged by the monopolist for the good being sold at store i. Store 1 is located at point $a \geq 0$ and store 2 at point $1-b \geq a$. This situation is depicted in figure 1 below.

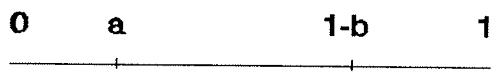

Figure 1

The demand functions for each of the two goods (or stores) are

$$y_1 = a + \frac{1-a-b}{2} + \frac{p_2 - p_1}{2t(1-a-b)}, \tag{2}$$

$$y_2 = 1 - y_1 = b + \frac{1-a-b}{2} + \frac{p_1 - p_2}{2t(1-a-b)}, \tag{3}$$

respectively. Assuming that unit costs are constant, and normalised to zero, the objective of the monopolist is then

$$\max_{a,b,p_{1,2}} \pi^M = p_1 y_1 + p_2 y_2. \tag{4}$$

The first order conditions w.r.t. prices are[1]

$$\frac{\delta\pi^M}{\delta p_1} = \frac{2p_1 - 2p_2 + a^2 t - b^2 t + 2bt - t}{2t(a+b-1)} = 0; \tag{5}$$

$$\frac{\delta\pi^M}{\delta p_2} = \frac{2p_2 - 2p_1 - a^2 t + b^2 t + 2at - t}{2t(a+b-1)} = 0. \tag{6}$$

Solving (5-6), we obtain

$$p_i = p_i - t(a+b-1); \tag{7}$$

condition (7) implies

$$p_1 = p_2; \quad a+b=1, \tag{8}$$

which amounts to saying that the monopolist has no incentive to differentiate. As pointed out by Tirole (1988, p.282), a social planner would choose a=b=1/4, since she aims at minimizing the transportation cost incurred by consumers. Then, as a first conclusion, we may say that the product variety offered by a monopolist is clearly socially suboptimal. The opposite conclusion holds for a duopoly, as it is well known (D'Aspremont, Gabszewicz and Thisse, 1979; Neven, 1985; Economides, 1986).

We haven't yet answered to the following question, though: what is the optimal

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1. The reader can easily verify that the first order conditions w.r.t. a and b are redundant.

price-and-location choice for the monopolist? As she doesn't differentiate, we can simply denote the location by a; obviously, given the simmetry of the problem, we imagine $0 \leq a \leq \frac{1}{2}$.[2] Since condition (1) must be satisfied, the highest price the monopolist can charge is

$$p^M = s - tx^2, \quad \Rightarrow \quad x \leq \sqrt{\frac{s}{t}}, \tag{9}$$

while the profit function is

$$\pi^M = (s - tx^2)y. \tag{10}$$

Notice that both a price-effect and a quantity-effect are present, of opposite sign. The demand function can be defined as $y=f(x)$, with

$$f(x) = 2x \quad \forall x \in]0, \frac{1}{2}]; \tag{11}$$

$$f(x) = 1 \quad \forall x \in \left[\frac{1}{2}, 1\right]. \tag{12}$$

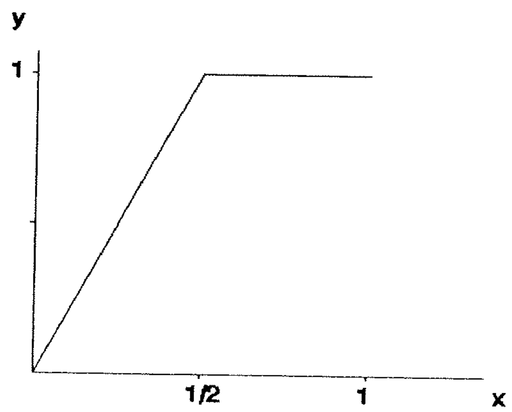

Figure 2

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2. For $\frac{1}{2} \leq a \leq 1$, see below.

The behaviour of *f(x)* is described in figure 2 above. The monopolist's objective function is:

$$\max_x \quad \pi^M = (s - tx^2)2x, \quad \forall x \in]0, \frac{1}{2}], \tag{13}$$

$$\max_x \quad \pi^M = (s - tx^2)1, \quad \forall x \in \left[\frac{1}{2}, 1\right]. \tag{14}$$

At $x = \frac{1}{2}$, (13) and (14) obviously coincide.

Let us first consider the objective function described by (13): the first order condition w.r.t. x is

$$\frac{\delta \pi^M}{\delta x} = 2s - 6tx^2 = 0, \tag{15}$$

which yields as a solution $x = \sqrt{\frac{s}{3t}}$;[3] given $x \in]0, \frac{1}{2}]$, this means that $\sqrt{\frac{s}{3t}} \leq \frac{1}{2}, \Rightarrow s \leq \frac{3}{4}t$. The equilibrium price is $p = \frac{2}{3}s$, while profit amounts to $\pi^M = \frac{4}{3}\sqrt{\frac{s^3}{3t}}$.

Let us now turn to (14). The of (14) w.r.t. x is

$$\frac{\delta \pi^M}{\delta x} = -2tx, \tag{16}$$

which is always negative over the relevant interval. This is because the quantity-effect dries up, while the price-effect is obviously negative. In other words, the maximum is located at $x = \frac{1}{2}$, and corresponds to $\pi^* = s - \frac{t}{4}$.

Thus, we must evaluate the following inequality

$$\pi^M\left(x = \frac{1}{2}\right) > \pi^M\left(x = \sqrt{\frac{s}{3t}}\right), \tag{17}$$

or

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3. This satisfies constraint (9) above, and implies the overall constraint $s \leq 3t$.

$$s - \frac{t}{4} > \frac{4}{3}\sqrt{\frac{s^3}{3t}}, \tag{18}$$

that is, we must evaluate the sign of the following expression

$$g(s,t) = 432s^2t + 27t^3 - 216st^2 - 256s^3; \tag{19}$$

In order to do this, we can resort to the equivalent formulation $\gamma(z) = \frac{g(s,t)}{t^3}$, with $z = \frac{s}{t}$. This yields

$$\gamma(z) = 432z^2 + 27 - 216z - 256z^3. \tag{20}$$

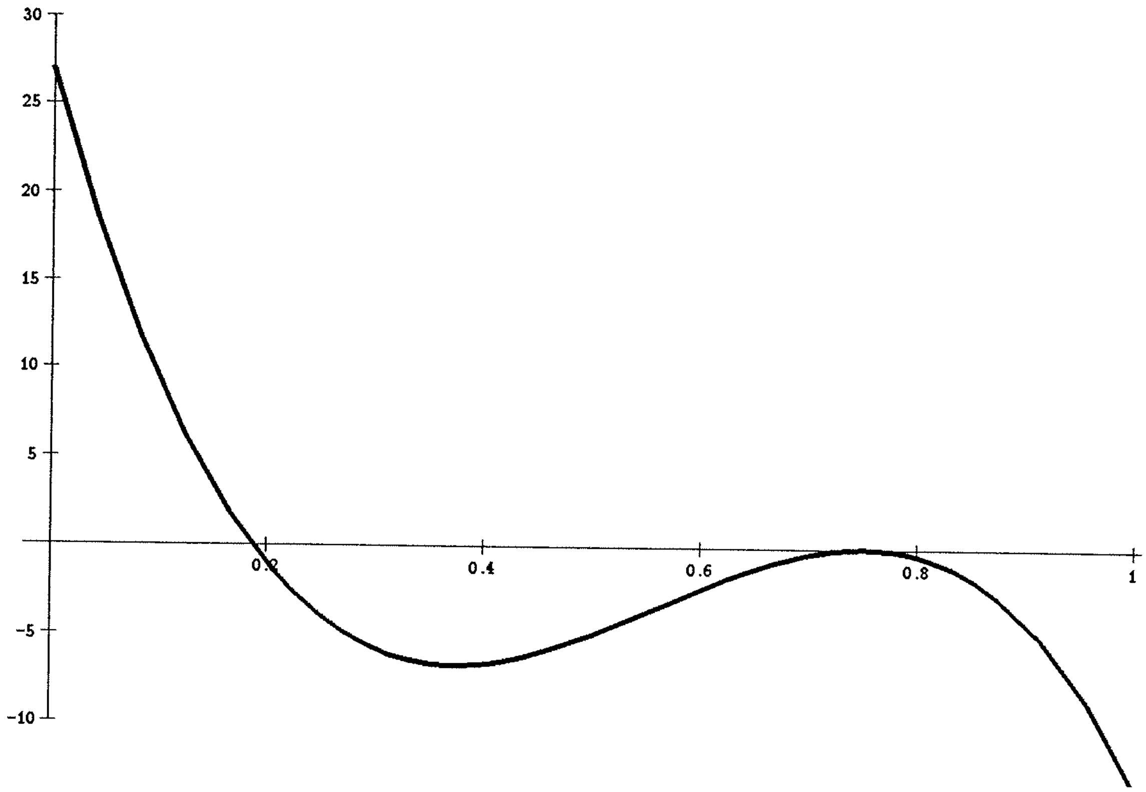

Figure 3

The shape of $\gamma(z)$ is depicted in figure 3 above.

The roots are $z_1 = \frac{3}{16}, z_2 = \frac{3}{4}, z_3 = \frac{3}{4}$.[4] The function is positive for $0 < z < \frac{3}{16}$; it is negative for $\frac{3}{16} < z < \frac{3}{4}$ and $z > \frac{3}{4}$,[5] while it goes to zero when $z = \frac{3}{16}$ and $z = \frac{s}{t} = \frac{3}{4}$. Obviously, $s - \frac{t}{4} > 0 \quad \forall \frac{s}{t} \in [0, \frac{1}{4}[$, so that our comparison makes sense for $\frac{s}{t} \in \left[\frac{1}{4}, \frac{3}{4}\right]$, since for $\frac{s}{t} < \frac{1}{4}$ the first term in (17) is negative, while for $\frac{s}{t} > \frac{3}{4}$ the relevant objective function is (14). Consequently, for $z = \frac{s}{t} = \frac{3}{4}$ the generic solution given by $\left(x = \sqrt{\frac{s}{3t}}; p^M = \frac{2}{3}s\right)$ and the corner solution given by $\left(x = \frac{1}{2}; p^M = s - \frac{t}{4}\right)$ obviously coincide, yielding a profit $\pi^M = \frac{t}{2}$.[6] If, on the other hand, $0 < s < \frac{3}{4}t$, the monopolist find it optimal to choose the inner solution, i.e., $x \in]0, \frac{1}{2}[$, the exact size of x being determined by the relative size of s and t. Finally, for $\frac{s}{t} > \frac{3}{4}$, given (16), the optimal solution is $x = \frac{1}{2}$.

Thus, we can formulate the monopolist's optimal price-and-location rule as follows: 'first, choose $x = \frac{1}{2}$ if $\frac{s}{t} \geq \frac{3}{4}$; choose $x = \sqrt{\frac{s}{3t}}$ if $\frac{s}{t} \in]0, \frac{3}{4}[$; then, choose $a=x$'.

For $\frac{1}{2} \leq a \leq 1$, the problem can be reformulated as follows:

$$\max_v \quad \pi^M = (s - tv^2)2v, \quad v = (1-x); \tag{21}$$

which yields

$$v = \sqrt{\frac{s}{3t}}, \quad x = 1 - \sqrt{\frac{s}{3t}}; \tag{22}$$

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4. As the reader can quickly verify, for both $z = \frac{3}{16}$ and $z = \frac{3}{4}$ the net utility accruing to marginal consumers is zero.

5. This range of values is excluded, since $x \in \left[0, \frac{1}{2}\right]$. See below.

6. Notice that this corresponds to the profit accruing to each duopolist under both the original linear transportation cost hypothesis adopted by Hotelling (1929) and the quadratic transportation cost one adopted by D'aspremont, Gabszewicz and Thisse (1979); cfr. Tirole (1988, p.280). Yet, these results are not comparable, since they have been derived from quite different models: in terms of ours, the equilibrium price charged under duopoly, $p_1=p_2=t$, is viable and allows the duopolists to satisfy a total demand $y_1+y_2=1$ if and only if $s \geq \frac{5}{4}t$. This can be verified by checking condition (1).

thus, we have $\pi^M = \frac{4}{3}\sqrt{\frac{s^3}{3t}}$, and we can easily verify that the sign of $\gamma(z)$ is negative for $z > \frac{3}{4}$, i.e., for $s > \frac{3}{4}t$. The result mirrors the one we derived previously, with the monopolist choosing the optimal location, $a=v$, starting from the right boundary of the city. This completes the analysis.

3. Conclusions

We tried to describe the optimal behaviour of a monopolist operating in a horizontally differentiated market. The outcome seems to be well summarized by the rule 'do not differentiate'. Furthermore, the optimal location, as well as the optimal pricing rule, depends on the relative size of the parameters defining consumer's surplus and transportation cost.

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