MATURITY EFFECTS OF FUTURES AND FORWARD PRICES IN A TWO-FACTOR GENERAL EQUILIBRIUM MODEL

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In the two-factor economy developed by Longstaff and Schwartz (1991) forward and futures prices of default-free bills and bonds are obtained and maturity effects analysed. It is shown that the relationship between futures price volatility and maturity is stochastic so that, as it may be seen through dynamic simulations over the 80s, the classical monotonic relation could sometimes be reversed.

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Longstaff and Schwartz (1991) (henceforth LS) have recently proposed a model in which asset prices are endogenously determined as function of both the short term interest rate and the volatility of its first differences. Their model is a specialization of the classical general equilibrium theory of Cox, Ingersoll and Ross (CIR) (1985a) to the case of two state variables X,Y described by two independent square-root (Feller-) diffusions affecting the production technology of the single consumption/investment good. By assumption the physical rate of return dQ(X,Y,t)/Q(X,Y,t) is governed by (omitting function arguments)

$$\frac{dQ}{Q}=(\mu X+\theta Y)\,dt+\sigma\sqrt{Y}\,dZ_1$$

where

$$dX=(a-bX)\,dt+c\sqrt{X}\,dZ_2$$

$$dY=(d-eY)\,dt+f\sqrt{Y}\,dZ_3$$

Assuming time-additive preferences and log-utility of consumption (as in CIR (1985b)) LS obtain the short-term interes rate r and the instantaneous variance of its changes $V=(dr)^2$ as linear functions of the state variables and therefore perfect substitutes for them:

$$r=\alpha X+\beta Y$$
$$V=\alpha^2 X+\beta^2 Y$$

In this economy, the fundamental partial differential equation (PDE) for the price H of an asset with coupon flow D and terminal condition $\theta(T)$ is given by

$$\frac{x}{2}H_{xx}+\frac{y}{2}H_{yy}+(\gamma-\delta x)H_x+(\eta-\xi y-\lambda y)H_y-rH+H_t+D=0$$
$$H(T)=\Theta(T) \tag{1}$$

where

$$x=X/c^2,\quad y=Y/f^2,\quad \gamma=a/c^2,\quad \delta=b,\quad \eta=d/f^2,\quad \xi=e$$

and λy is the endogenous market price of risk ([1]).
For default-free unit zero-coupon bonds maturing at time T the PDE is identified by D=0 and $\theta(T)=1$. Let B(t,T) be the solution price (see LS equation (20)). In the following we shall analyse futures and forward contracts written on default-free bonds.

I. Futures and forward prices with stochastic interest rates and volatility.

A forward contract is an agreement to exchange, at a future (delivery) date S, one unit of the underlying asset (vehicle) with a prespecified amount H^f (the forward price). In our economy, the difference between futures and forward contracts stems from the marking-to-market process through which any futures contract is rewritten at the end of each "trading day" at the closing futures prices H^f, the difference between the new price and the previous (contractual) one being paied to the long investor by the investor in the short position.

Using arbitrage arguments (CIR (1981)) it can be shown that the time t futures price of a zero-coupon bond maturing at T with delivery date S<T solves (1) with D=0 and

$$\Theta(S) = B(S,T)\exp\left(\int_t^S r(u)\,du\right)$$

or, equivalently, D=rH and $\theta(S)=B(S,T)$.

Using standard techniques for PDE it can be shown that the solving futures price $H^f(r,V,t,S,T)$ is given, in closed form, by

$$H^f = F_1(S,T)M_1(t,S,T)G_1(S,T)N_1(t,S,T)$$
$$.\exp\left(\frac{\alpha^2 N_2(t,S,T)-\beta^2 M_2(t,S,T)}{\alpha\beta(\beta-\alpha)}r+\frac{\beta M_2(t,S,T)-\alpha N_2(t,S,T)}{\alpha\beta(\beta-\alpha)}V\right)$$

where

$$F_1(S,T)=\left[\frac{2\phi\exp((\phi+\delta)(T-S)/2)}{(\phi+\delta)(\exp(\phi(T-S))-1)+2\phi}\right]^{2\gamma}$$

$$M_1(t,S,T)=\left[\frac{2\delta}{F_2(S,T)(1-\exp(-\delta(S-t)))+2\delta}\right]^{2\gamma}$$

$$G_1(S,T)=\left[\frac{2\psi\exp((\psi+(\xi+\lambda))(T-S)/2)}{(\psi+(\xi+\lambda))(\exp(\psi(T-S))-1)+2\psi}\right]^{2\eta}$$

$$N_1(t,S,T)=\left[\frac{2(\xi+\lambda)}{G_2(S,T)(1-\exp(-(\xi+\lambda)(S-t)))+2(\xi+\lambda)}\right]^{2\eta}$$

$$\phi=\sqrt{2\alpha+\delta^2}$$

$$\psi=\sqrt{2\beta+(\xi+\lambda)^2}$$

and

$$F_2(S,T)=\frac{2\alpha(\exp(\phi(T-S))-1)}{(\phi+\delta)(\exp(\phi(T-S))-1)+2\phi}$$

$$M_2(t,S,T)=\frac{2\delta F_2(S,T)\exp(-\delta(S-t))}{F_2(S,T)(1-\exp(-\delta(S-T)))+2\delta}$$

$$G_2(S,T)=\frac{2\beta(\exp(\psi(T-S))-1)}{(\psi+(\xi+\lambda))(\exp(\psi(T-S))-1)+2\psi}$$

$$N_2(t,S,T)=\frac{2(\xi+\lambda)G_2(S,T)\exp(-(\xi+\lambda)(S-t))}{G_2(S,T)(1-\exp(-(\xi+\lambda)(S-t)))+2(\xi+\lambda)}$$

Let ρ_t be the present value operator (Ross (1978)). Through a simple arbitrage argument ([2]) it can be shown that the forward price $H^f(r,V,t,S,T)$ for delivery in S of a zero-coupon bond maturing at T (>S) must satisfy the relation

$$H^f\, B(r,V,t,S) = \rho_t(1(T)) = B(r,V,t,T)$$

$B(r,V,t,T)$ being the price of one ('money') unit at T:

$$B(r,V,t,T) = F_1(t,T)\, G_1(t,T) \exp\Big(\frac{\alpha^2 G_2(t,T) - \beta^2 F_2(t,T)}{\alpha\beta(\beta-\alpha)} r + \frac{\beta F_2(t,T) - \alpha G_2(t,T)}{\alpha\beta(\beta-\alpha)} V\Big)$$

In the case of a coupon bond maturing at T with coupon payments (c_h, h=1,..,n) we have, by linearity and the chain property of the present value operator,

$$H^F B(t,S) = \rho_t\Big(\sum_1^n c_h B(S,S+h)\Big) =$$

$$\sum_1^n c_h \rho_t(\rho_S(1(S+h)) = \sum_1^n c_h B(t,S+h) = C(t,c_h,T)$$

where T=S+n, $C(t,c_h,T)$ is the current price of the coupon bond and 1(S+h) is one money unit at time S+h.

A simular result holds for futures on bonds. In fact

$$H^f = \rho_t\Big(C(S,c_h,T) \exp\Big(\int_t^S r(u)\,du\Big)\Big) =$$

$$\sum_1^n c_h \rho_t\Big(B(S,S+h) \exp\Big(\int_t^S r(u)\,du\Big)\Big) = \sum_1^n c_h H^f(t,S,S+h)$$

II. Futures and forward price volatilities and the maturity effect.

According to Samuelson's (1965) hypothesis the volatility of futures prices increases as the delivery date approaches. This 'maturity effect' was found to be in large agreement with real world futures prices by a number of studies ([3]).

Up to now, however, no analysis is available in the case of stochastic interest rates, even if this is the general situation in which investors operate and, as a rule, there is a presumption of relevant effects of a stochastic environment with respect to a deterministic one ([4]). The LS economy seems an adequate framework to implement this analysis.

Given that in this model asset prices are diffusions processes, it can be shown that the variance per unit time of forward and futures is

$$\left(\frac{dH^F}{H^F}\right)^2 = \frac{\beta^2[F_2(t,T)-F_2(t,S)]^2-\alpha^2[G_2(t,T)-G_2(t,S)]^2}{\alpha\beta(\beta-\alpha)}r + \frac{\alpha[G_2(t,T)-G_2(t,S)]^2-\beta[F_2(t,T)-F_2(t,S)]^2}{\alpha\beta(\beta-\alpha)}V \qquad (2)$$

and

$$\left(\frac{dH^f}{H^f}\right)^2 = \frac{\beta^2 M_2(t,S,T)^2-\alpha^2 N_2(t,S,T)^2}{\alpha\beta(\beta-\alpha)}r + \frac{\alpha N_2(t,S,T)^2-\beta M_2(t,S,T)^2}{\alpha\beta(\beta-\alpha)}V \qquad (3)$$

so that, with stochastic interest rates, the maturity effect becomes a stochastic relationship.

This analytical results can be graphically explained.

Let us compare first the futures and forward volatilities across contract maturities.

In Fig. 1 ([5]) it is shown both the increasing volatility of forward and futures prices as the number of periods to delivery goes to zero and the greater variability of futures with respect to forward prices. The first result is an effect of the equivalence at maturity among forward, futures and spot prices of the asset on which the contracts were written (a 2 years T-Bill). This clearly implies the convergence of their volatilities. The second result is similar to the greater variability of short term interest rates with respect to interest rates for longer maturities. In fact, by the marking-to-market process, the futures contract is analogous to rolling over short positions whilst a forward contract is similar to a going-long strategy.

Using actual data on U.S. T-bill rates ('87-'89) to simulate the model predictions, it can be seen that the forward and futures volatilities increase, with some noise, up to the spot volatility prevailing at maturity (Fig. 2).

In Fig. 3 for a given time to delivery (1 year), forward and futures volatilities are cross-sectionally compared as the vehicle maturity increases (from 1 to 120 months).

Similar results can be obtained in terms of price (semi-) elasticities, defined by $-H_r/H$, $-H_V/H$.

The Samuelson's maturity effect concerns, instead, the temporal behaviour of forward and futures prices of given contracts. Considering the period '85-'89 in which, by hypothesis, a forward and a futures contract were written at the beginning and closed at the end, we obtain the price changes depicted in Fig. 4, evidencing the increasing volatility of forward and futures as the delivery date approaches. The effect is slightly greater for futures prices and greater the greater the vehicle maturity (Fig. 5). As we have shown, however, the relationship between futures price variability and time to maturity is non deterministic. Depending on the state variable dynamics prevailing in the period it may be increasing or decreasing or neither of them. This is the meaning of the simulation in Fig. 6, concerning the years '79-'84 of high and variable interest rates: the futures volatility during the last year before maturity is found to be substantially smaller than in earlier period of the contract lifetime.

Conclusions

In the two-factor general equilibrium model of Longstaff and Schwartz (1991) we have obtained futures and forward prices of Treasury bills and bonds. These prices are used to show that in the case of stochastic interest rates a monotonic relation between futures price volatility and maturity no longer exists. Cross-section and cross-time simulations using historical data on a period of high interest rates and volatility graphically explain how the maturity effect could sometimes be reversed.

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FOOTNOTES

1. The state variable X is not a risk factor given that, by assumption, its changes are orthogonal to changes in physical production and individuals obtain utility only from real consumption.

2. Write, at time t, a forward contract on a T-period bill with forward price H^f and buy at the same time H^f zero-coupon unit discount bonds maturing at S. At time S you will receive H^f money units to be exchanged with the bond the forward contract was written on.

3. See for example Anderson (1985), Milonas (1986) and their references. The evidence, however, does not always support the maturity effect: Milonas (1986) rejects the hypothesis for the corn market during 1972-1983; Ball and Torous (1986) for the case of gold futures 1983-1984.

4. Nonstochastic interest rates are used in Ball and Torous (1986) and Ross (1989).

5. The parameter estimates are obtained from monthly data (june 1964 - december 1989) on the one-month U.S. Treasury-bill yield and are given by (see LS): $\alpha=0.0957$, $\beta=0.1889$, $\gamma=0.02211$, $\delta=0.3241$, $\eta=0.0122$, $\nu=\xi+\lambda=3.0192$. γ and η are obtained from LS equations (16)-(19). Static simulations use the sample means r=0.0672 and V=0.0001. Dynamic simulations use tha actual data set.

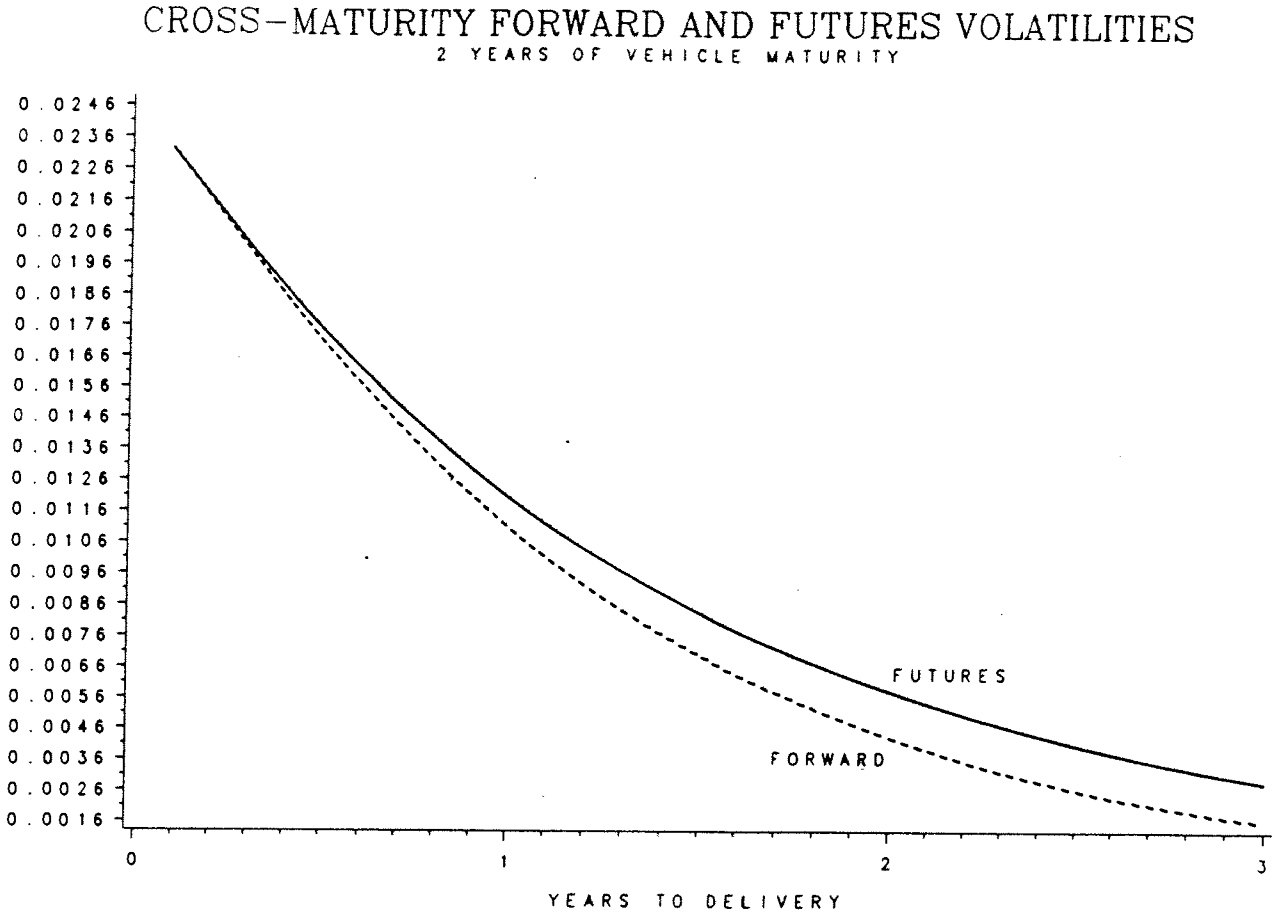

Figure 1. Cross-maturity forward and futures volatilities. The parameters estimates are obtained from monthly data (june 1964 - december 1989) on the one-month U.S. Treasury-bill yield and are given by: α=0.0957, β=0.1889, γ=0.02211, δ=0.3241, η=0.0122, $\nu=\xi+\lambda$=3.0192. The short term interest rate and the volatility take the sample means r=0.0672 and V=0.0001. The vehicle maturity is two years.

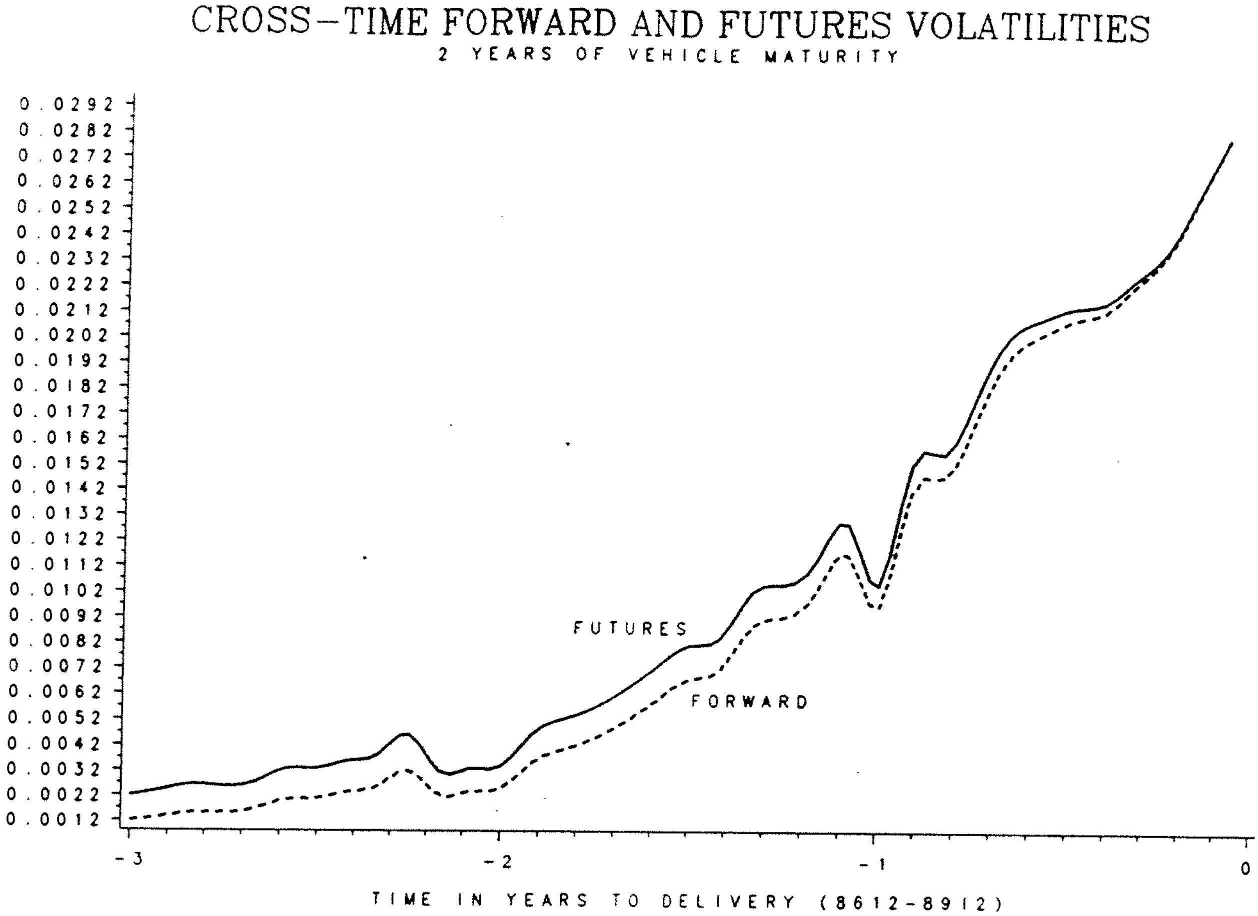

Figure 2. Cross-time forward and futures volatilities. The parameters estimates are obtained from monthly data (june 1964 - december 1989) on the one-month U.S. Treasury-bill yield and are given by: α=0.0957, β=0.1889, γ=0.02211, δ=0.3241, η=0.0122, $\nu=\xi+\lambda$=3.0192. The time axis represents december 1986 - december 1989. The vehicle maturity is two years.

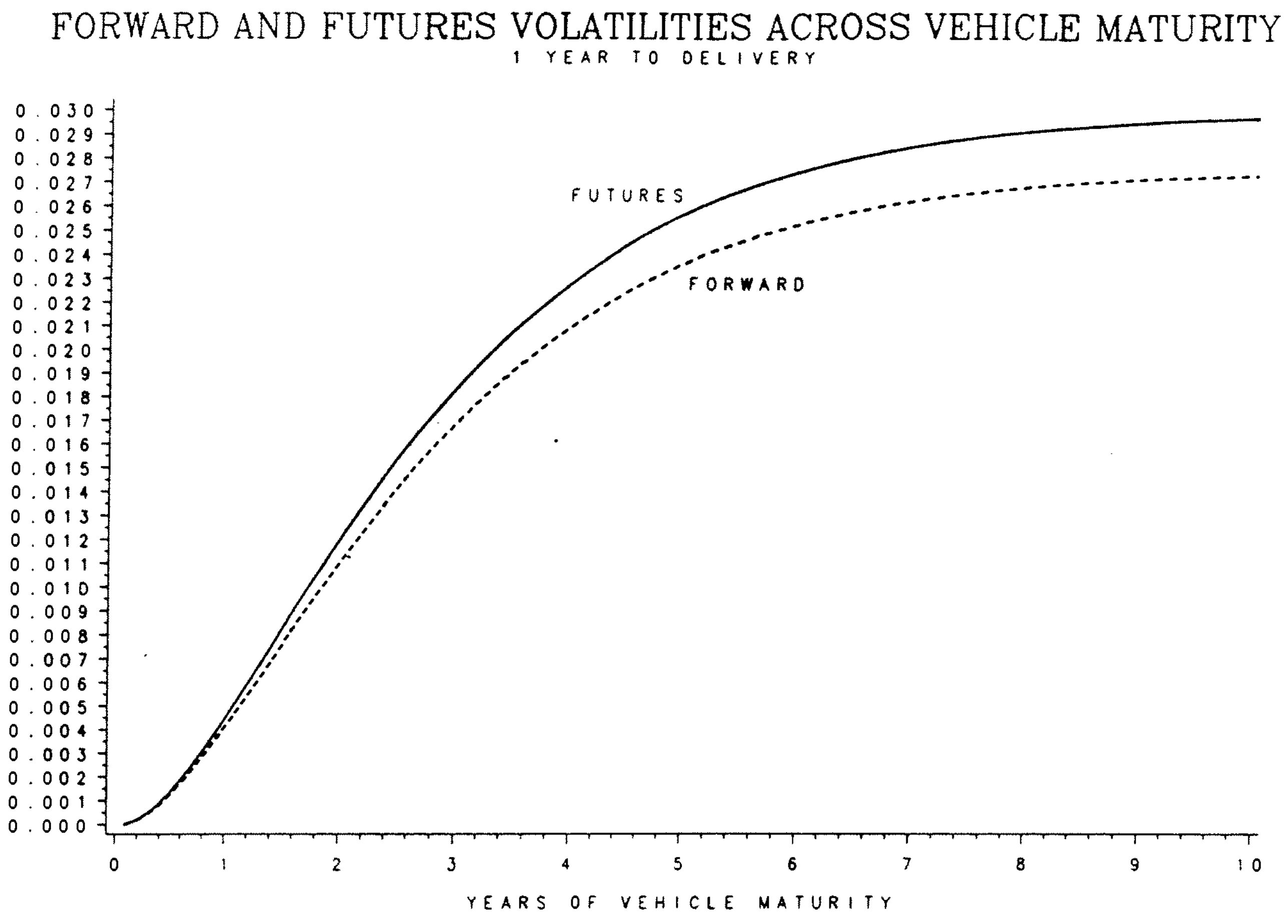

Figure 3. Forward and futures volatilities across vehicle maturity.

The parameters estimates are obtained from monthly data (june 1964 - december 1989) on the one-month U.S. Treasury-bill yield and are given by: α=0.0957, β=0.1889, γ=0.02211, δ=0.3241, η=0.0122, $\nu=\xi+\lambda$=3.0192. The short term interest rate and the volatility take the sample means r=0.0672 and V=0.0001. The time to delivery is one year.

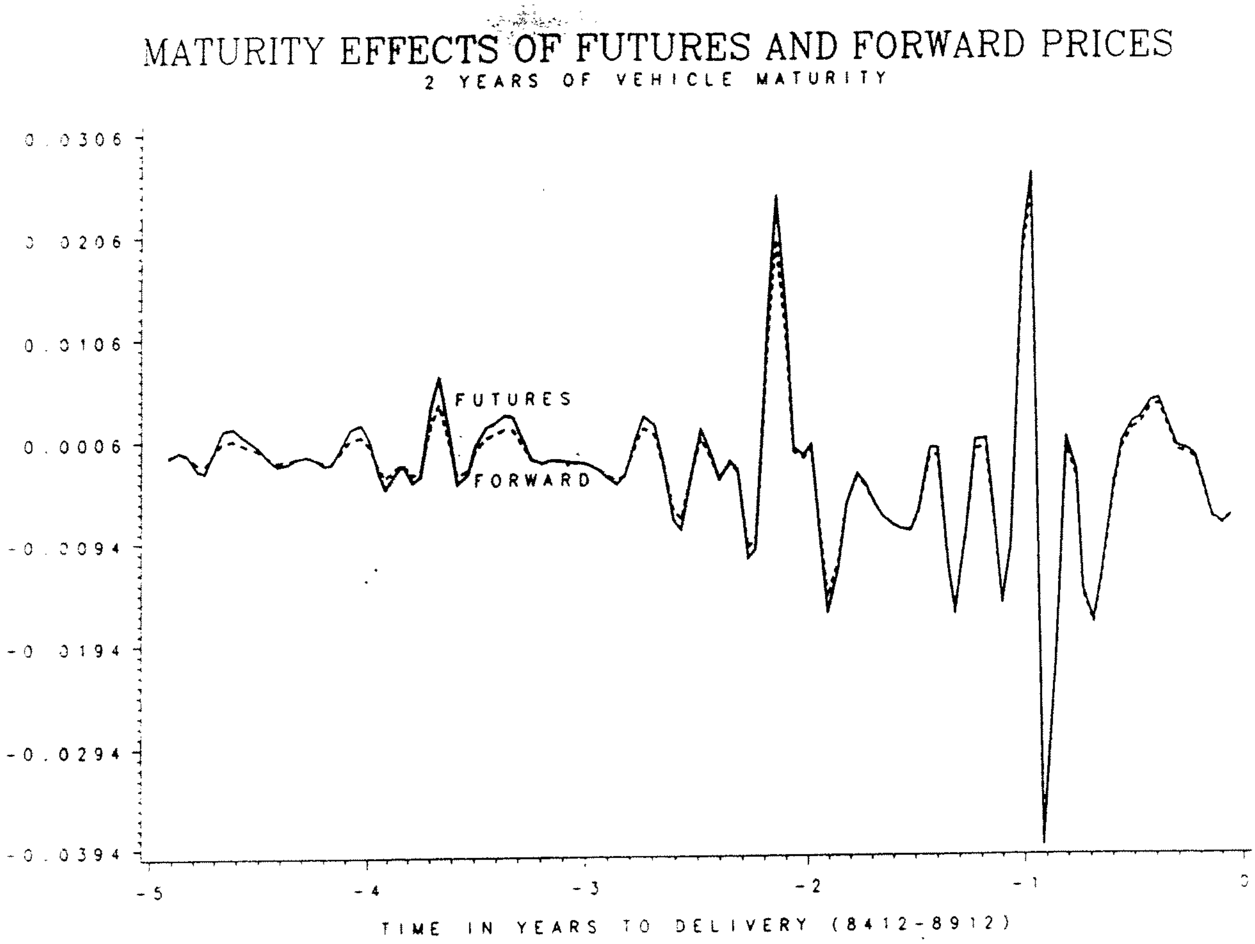

Figure 4. Maturity effects of futures and forward prices. The parameters estimates are obtained from monthly data (june 1964 - december 1989) on the one-month U.S. Treasury-bill yield and are given by: α=0.0957, β=0.1889, γ=0.02211, δ=0.3241, η=0.0122, $\nu=\xi+\lambda$=3.0192. The time axis represents december 1984 - december 1989. The vertical axis represents price changes. The vehicle maturity is two years.

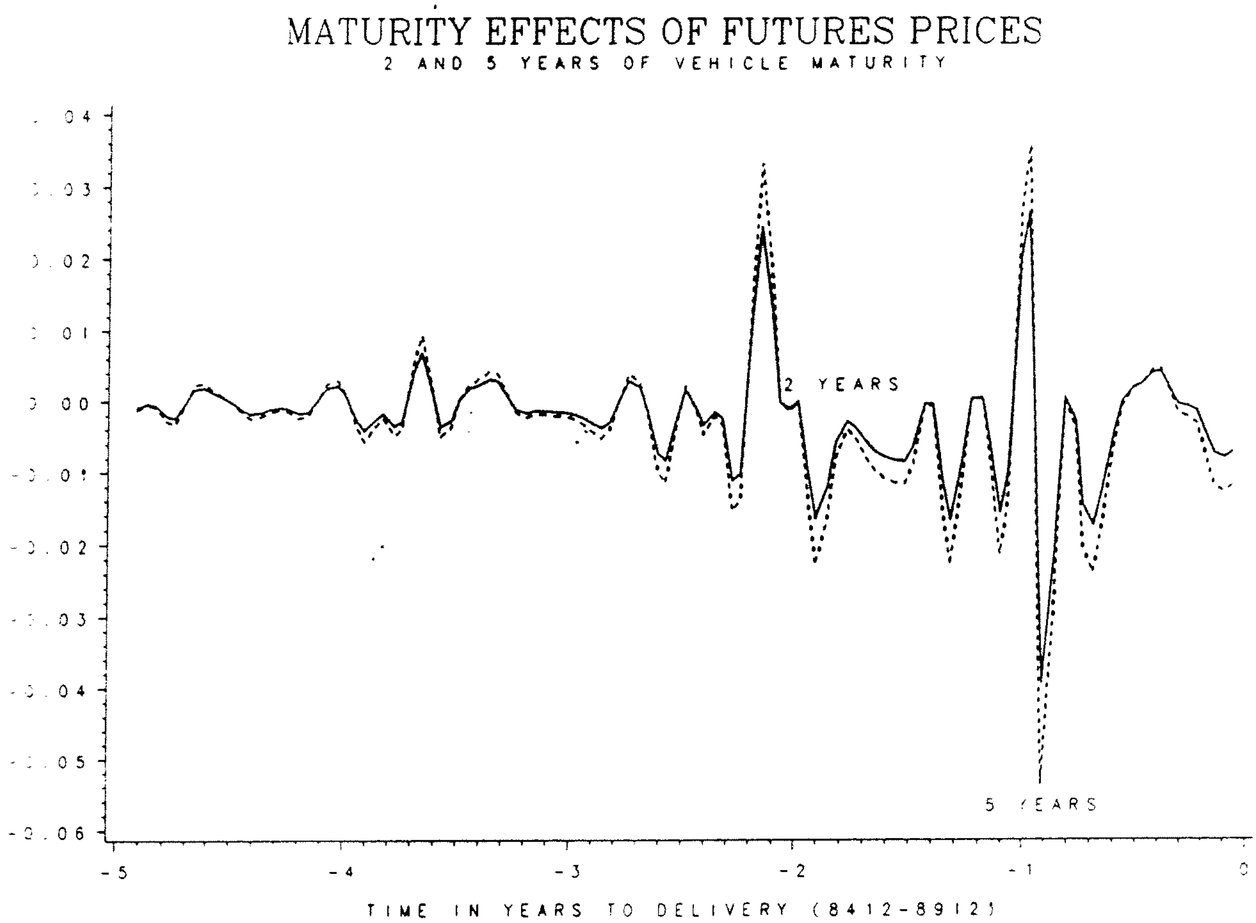

Figure 5. Maturity effects of futures prices.

The parameters estimates are obtained from monthly data (june 1964 - december 1989) on the one-month U.S. Treasury-bill yield and are given by: α=0.0957, β=0.1889, γ=0.02211, δ=0.3241, η=0.0122, $\nu=\xi+\lambda$=3.0192. The time axis represents december 1984 - december 1989. The vertical axis represents price changes. The vehicle maturities are two and five years.

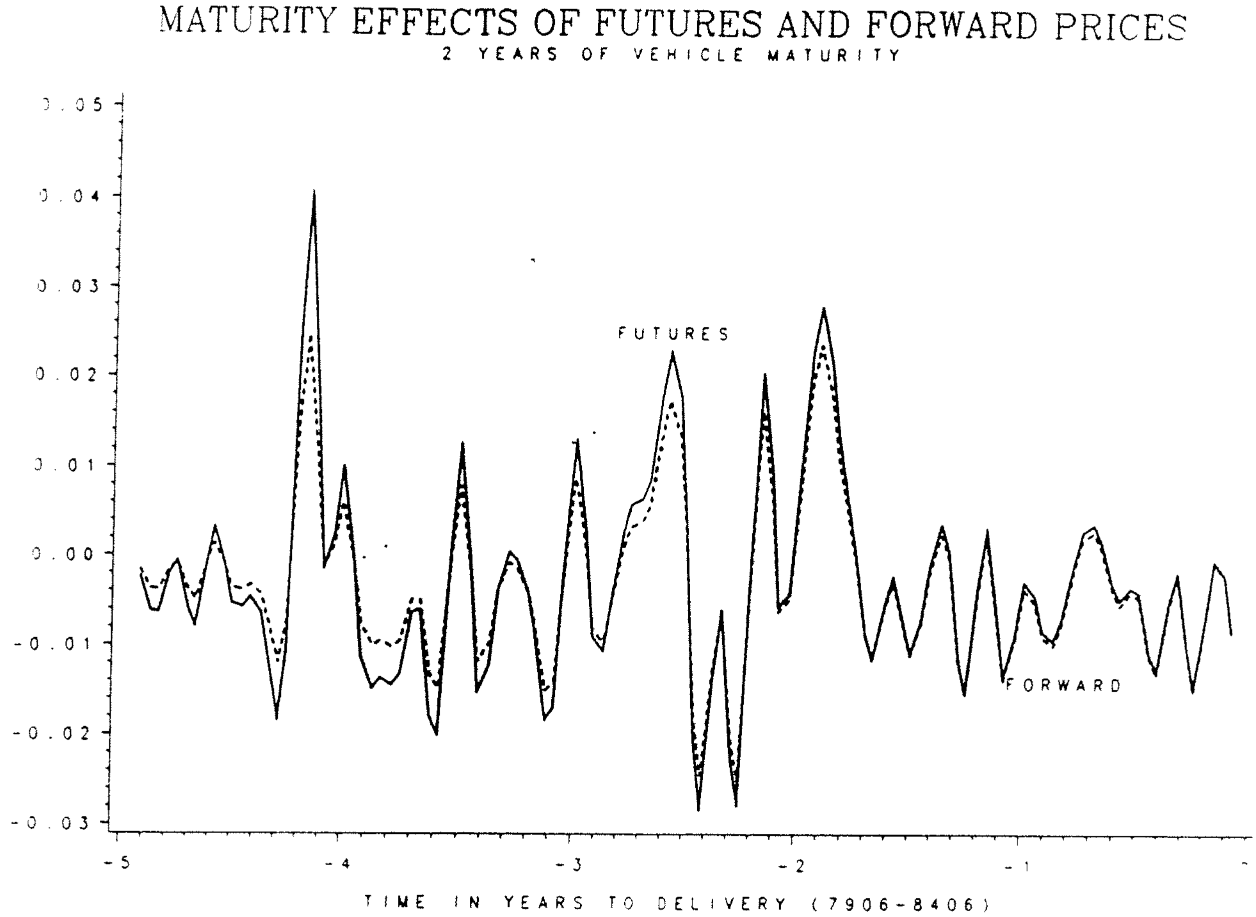

Figure 6. Maturity effects of futures and forward prices. The parameters estimates are obtained from monthly data (june 1964 - december 1989) on the one-month U.S. Treasury-bill yield and are given by: α=0.0957, β=0.1889, γ=0.02211, δ=0.3241, η=0.0122, $\nu=\xi+\lambda$=3.0192. The time axis represents june 1979 - june 1984. The vertical axis represents price changes. The vehicle maturity is two years.