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DIRECT AND INDIRECT UTILITY FUNCTIONS UNDER UNCERTAINTY

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ABSTRACT

Some aspects of decision taking under risk are examined. The existence of fair gambles whose price equals their expected values is questioned and in addition the origin of expected utility is analysed. Then a diverse approach to decision taking under risk is considered, whereby agents maximize a utility function of consequences of actions. Consequences are treated like different goods and a prospect is like a bundle of goods. Stochastic dominance is invoked as a second stage criterion to decide whether to accept or not a prospect. Even though some problems arise as to the continuity by using the indirect utility function we are able to explain the Allais Paradox.

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GIANPAOLO ROSSINI

1. INTRODUCTION

Recent literature has deeply critizised the predective ability of Von Neumann Morgenstern expected utility model of decision making under uncertainty, while keeping its normative validity.

Expected utility and its fairly famous antecedents (Bernoulli, 1738) have all been able to get out of the dead end in which games of an infinite actuarial value should call for infinite prices.

Since nobody will ever bet an infinite certain amount of money in exchange for an uncertain infinite amount, to avoid an apparent absurdity it seemed reasonable to bound bet prices by introducing concave utility functions which were attached to all possible cosequence of any decision.

Most of the problems today are linked to the way utilities of consequnces are assembled together.

It seems to me that some of these hindrances scattered around the ground of decision theory could be avoided

1. once we recognize that nobody is going to offer you any infinite amount of money even though you bet it entirely,
2. when we go back to the utility functions of consequences.

2.GAMES AGAINST NATURE: AN EMPTY TAXONOMY

The expected utility criterion has been devised when the problem of finding an equilibrium arose in games which did not have one in pure strategies. Expected utility was not thought of as a solution of the St.Petersburg paradox.Indeed when risk neutrality and risk loving are displayed the St.Petersburg paradox is back even though we adopt a Von Neumann Morgenstern utility function .
The context in which the expected utility model was born is relevant not only for that. This context should be reconsidered also by those who include expected utility among the criteria for decision taking in so called "games against nature". Its neglect seems to be one of the reason why the expected utility model has been sometimes incorrectly used. Let me be more explicit.
When I consider a game against nature the consequences of my actions are determined just by the state of nature which is going to come true. Nature is not deemed to be a supplier of states in any economic sense.In other words nature is not an agent offering bets, yet it is simply an element which lies beyond economic control by any agent.
However St.Petersburg paradox cannot be considered as a game against nature. Yet if that is not the case I have to ask simply who in St.Petersburg paradox is going to offer an infinite amount of money for a bet which could cost an infinite amount of money (see Machina ,1982).
This is to say that every time we consider some economic decision we cannot consider just nature (in the form of chance) yet also

a market where somebody supplies bets (nature does supply bets but they are all free goods). Since risk neutrality and risk loving on the supply side of the market would throw us back to the moving sands of St.Petersburg paradox we are led to imagine bet suppliers who are risk averse.

3.RISK AVERSE BET SUPPLIERS

When risk aversion casts its shadow on both sides of a market for bets, equilibrium can be reached

1.either if we have different (asymmetric) information by which one knows more than the other and has also different subjective probabilities

2.or if the agents have different utility functions of consequences.

These two situations could be well depicted by a diagram like that in figure 1 where risk averse suppliers and demanders meet.In figure 1 we have a diagram in which I have plotted a supply and a demand curve on the assumption that: a)suppliers have an initial information advantage which allows them to price bet lower b)suppliers are more risk averse.The diagram has some point in common with Tobin (Tobin,1958-1959, fig.3.1,pg.73),yet the y axis represents the risk premium instead of expected values and the origin does not correspond to a zero risk value.
If the two assumption above are satisfied an equilibrium in the market can be obtained, while in all other cases there may be unstability or even no equilibrium possible.

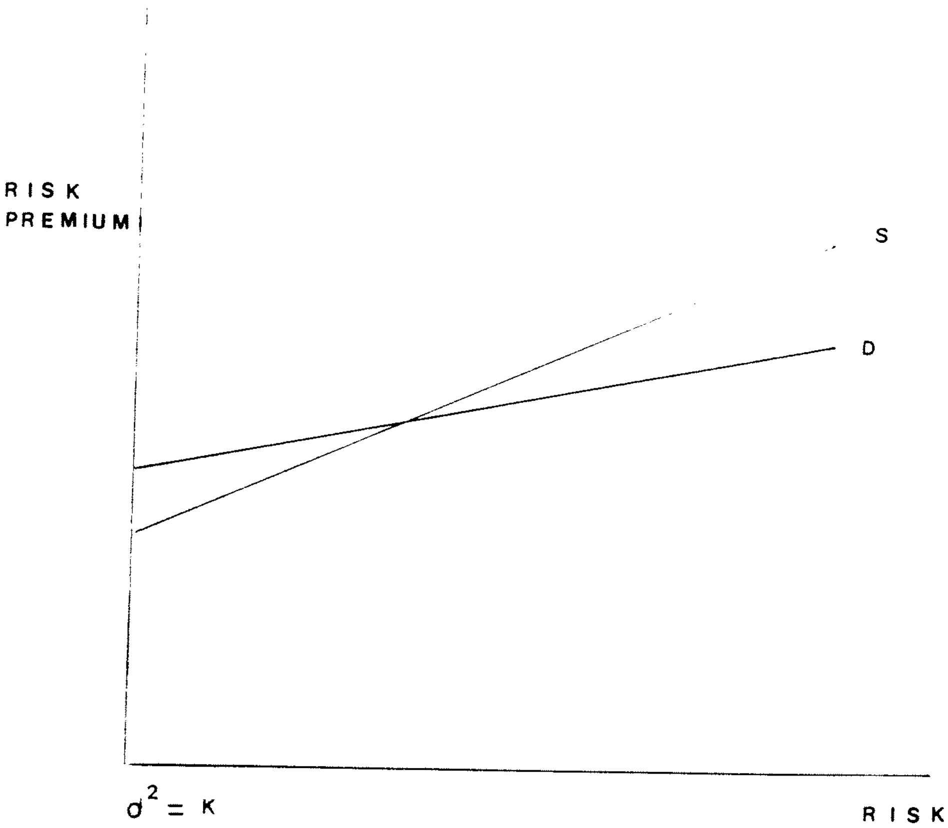
fig.1

Is there actually any apparent reason why I should so strongly emphasize the role of bet suppliers? The reason should become clear quite soon after having introduced to our new approach of decision making under risk.

4. A SIMPLE UTILITY APPROACH

Now we would like to sketch a simple way to model a demand for "uncertain ventures" instead of limiting ourself to the reexamination of expected utility.

It seems to me that one of the peculiar feature of Von Neumann Morgenstern expected utility model is the argument of the utility functions of consequences (usually called in the literature U(.) to distinguish it from V(.) relative to the utility of a prospect) . This argument appears expressed as different amounts of money.



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(or at least of only one good which does not change over the states) relative to different states of the world .

Since there are not complete market for all states of the world I cannot think of different amounts of money as different goods associated to different contingencies. Yet we must resort to probabilities to evaluate each state of the world. In this sense probabilities may appear as shadow prices of states of the world which could even differ across individuals. To this purpose choices can be expressed in a way which may appear fairly unusual. I assume that each agent possesses a utility function over the consequences of a kind which is reminiscent of consumer choice under certainty. Every consequence is treated like a good and a prospect is a bundle of these goods.

How does the consumer make his choice?

Fierst of all he must have a budget set to make his or her problem determined. The budget set is determined by introducing probabilities of states as prices and making the expected value of a given prospect the constraint. It will then be possible to derive some sort of demand functions for money prizes in each state. Yet how shall I be able to say whether an agent is willing to enter a lottery or an uncertain venture?

The answer is that an agent will buy the ticket when the prospect formed by his demands and by the probabilities is stochatically dominated by the one supplied by the market or at least equivalent (see Rothschild-Stiglitz, 1970). This gives rise to modal choices at the individual level.

More precisely we can write:

PROPOSITION 1

<u>Each agent possesses a utility function over bundles of consequences. He decides to buy a prospect iff this dominates stochastically his optimal prospect or is at least equivalent</u>.

The consumer derives his optimal prospect by maximizing

$U\ (x_1, x_2, \ldots, x_n)$ (1)

subject to

$\Sigma_i p_i x_i = E \qquad i = 1.....n$ (2)

where p_i are the probabilities which could or could not coincide with those offered by the bet suppliers; E is the expected value of the lottery .

We can distinguish two cases : the <u>first</u> when the probabilities of the prospect offered do not coincide with those of the consumer and the <u>second</u> when the probabilities coincide.

In the <u>first</u> case p_i in (2) are those of the consumer who will maximise (1) obtaining the optimal quantities x_i at the probabilities p_i that I call $x^*|p$. Once this operation has been performed the consumer compares his optimal prospect with the one offered whose expected value is E. He will buy the bet only if the one offered dominates or at least is stochastically equivalent to his optimal one.

In the <u>second</u> case the procedure is the same yet the comparison becomes more easy since with equal probabilities it is extremely simple to see which prospect dominates the other.

Here is an <u>example</u> based on a simple Cobb-Douglas utility function and a two consequence prospect.

A prospect is offered giving 10 ecu with probability .4 and 20 with probability .6 making the expected value of the prospect 16. The consumer is assumed to possess a CD of the form

$U(x_1,x_2)= x_1^{.2}\ x_2^{.8}$

the optimal quantities derived are

$x_1 =.2\ E/.4 = 8$

$x_2 =.8\ E/.6 = 21.3$

It is easy to see that the bet offered stochastically dominates the one demanded at those probabilities (simply observe that it has a greater variance which is a necessary condition for stochastic dominance).

This means that the consumer will enter that venture .

5.A LESS SIMPLE UTILITY APPROACH

There is another avenue based on the indirect utility function which could be considered. Usually the undirect utility function (iuf) is expressed in terms of prices and income and has some properties on which we shall not dwell now (see Deaton-Muellbauer,1980;Diewert,1974).

This function - when the direct utility function is of the CD type and when prices are thought to be the probabilities and the budget set is limited by the expected value of the prospect- is as follows:

$$V(p_1,p_2,E) = \ln E - a \ln p_1 - (1 - a) \ln p_2 \qquad (3)$$

as before the parameter a indicates the degree of substitution among prizes. The iuf gives the maximum utility for given probabilities and for a given expected value of the prospect.
The Roy's identity tells us that the optimal level of each prize will be given by

$$x_{1,2}(p_{1,2},E) = - [\delta v(\mathbf{p},E)/\delta p_{1,2}]/[\delta(\mathbf{p},E)/\delta E] \qquad (4)$$

The problem is that to have this nice duality result and to be able to get an indirect utility function we have to analyse the direct utility function and then to see whether the non standard utility function depicted in (1) can generate the one in (3) and see what are the properties.

Now it should again be emphasized that the demand functions derived with the Roy's identity have to pass the same scrutiny as before to see whether what is offered dominates or at least is equivalent to what is demanded.

6.THE DIRECT UTILITY FUNCTION:DERIVATION

It does not appear so easy to ground the decision criterion which can be derived by using the direct utility function under uncertainty.

In order to derive from preferences a utility function we have to define the set of possible consequences (in consumer theory this corresponds to the consumption set). We have to consider this set as the nonnegative orthant in R^n. This does not mean that we cannot include losses since we consider the domain of the utility function made by final states when each final state represents initial wealth plus the payoffs (the set F) [see the affine approach introduced by Kahneman-Tversky (1979)].

Then we assume:

1. Completeness or general comparability : an agent is able to compare any pair of final states $x,y \in F$ showing preference or indifference between any two.

2. Transitivity for any $x,y,z \in F$ we have that if x P y and y P z then x P z where P means weakly preferred.

3. Continuity: for each $x \in F$ [x:x Py] and [x: y P x] are closed sets.

4. Non satiation for each $x,y \in F$ if $x \geq y$ then x P y

PROPOSITION 2

<u>Whenever preferences obey assumptions 1-4 we can have a utility function from R^n to R which is continuous</u> .

For the proof we can resort to a standard one like that given by Varian (1984).

What makes our treatment peculiar is the way we model the

decision of an agent. Even though the utility function is a well behaved one the budget set is built in a non traditional way and the decision is a <u>modal</u> one.
Is there any sense in that and what are the aims of that? Let us try to answer to either question.

<u>Firstly</u> by using this procedure we can take into account the fact that bet supply has to be considered as well. This is relevant since only certain sorts of bet supplies may entail fair bets.

<u>Secondly</u> risk aversion can be evaluated by comparing on both sides of the market the value of offers and demand with the expected value of the bet. This seems to give rise to some sort of risk measure which can be derived without any knowledge of utility function. Let us see how we can do that.
To measure the risk aversion of an agent we have to compare the variance of the prospect as it is objectively given with the variance of the prospect demanded. Since the variance is a statistic which is influenced by the unit of measure of the observations we must deflate it to get a standard free index.
Suppose the variance of the prospect is V_p and that the variance of the demanded bet is V_d, then our measure becomes

$$\frac{V_d - V_p}{(V_p + V_d)/2} \qquad (5)$$

This can be thought of as a measure of risk aversion if the prospect suppliers are neutral to risk. In this case calculation is very easy.

A more difficult task has to be undertaken if bet suppliers are risk averse,while for risk loving it looks fairly simple.

In the former case the prospect demanded should be the result of a maximization process where the budget constraint is no longer represented by the expected value of the prospect but by the expected value of the prospect offered.In this case those who produce the bets ask for an entrance premium which should be paid by those who enter the risky venture.Take for instance the case of a car insurance policy.You pay a premium partly as a coverage of the pure risk you incur by driving your car and partly as a ticket to enter the insurance game. Indeed you can bet almost as much as you like. For instance you can buy life insurances of almost any amount even if the objective expected value of your life is far lower and not as much variable;you can buy car insurances with or without deductible and with different levels of maximum coverage.

Take again the case of a life insurance policy whose value is ten times higher than the expected value of the life insured. It seems obvious that this sort of insurance could give rise to behaviour of the beneficiaries characterised by a great degree of moral (or perhaps immoral) hazard by relatives. In that case the insurance company has to be risk averse and to let the insured person to pay a twofold premium:

one for avoiding the risk and

another to enter the game,since in that case the insurance company would be quite irrational to offer a fair gamble.

In this case risk aversion should be measured by the same method above by considering V_p the variance of the prospect and not .

the variance of the offer.

7.THE ALLAIS PARADOX OR CERTAINTY EFFECT

Maurice Allais (1953) analysed the expected utility model and presented a series of choice problems in which observed preferences of the agents did not obey the axiom of independence of Von Neumann Morgenstern expected utility.

One of the various choice problems in which people involved tend to violate the axioms is one in which there is an event, which is almost certain to be compared to probable ones. In these cases people seem to display a certainty preference from which the label certainty effect.

Let us consider a pair of choice problems proposed by Kahneman-Tversky (1979) to illustrate the so called Allais paradox.

Consider the first choice problem:

Prospect A

2500 ecu with probability .33

2400 .. .. .66

0 .. .. .01

Prospect B

2400 ecu with certainty

Consider a second choice problem

Prospect C

2500 ecu with probability .33

0 .. .. .67

Prospect D

2400 ecu with probability .34

0 .. .. .66

In practical experiments conducted in Israel by Kahneman-Tversky it was found the following pattern of preferences displayed by the majority of agents involved:

B P A and C P D

Now let us see why this choice apparently violates expected utility.

If u(0) = 0 we have that B P A implies:

$$u(2400) > .33u(2500) + .66u(2400) \quad (6)$$

because the expected utility is linear in probabilities we can write the inequality (6) as:

$$.34u(2400) > .33u(2500) \quad (7)$$

Yet this is just the opposite of what displayed in the choice between C and D when the result was:

$$.33u(2500) > .34u(2400) \quad (8)$$

which is the opposite of (7).

This is an apparent contraddiction whenever we want to explain actual choices in terms of the expected utility model.We limit to this case even though the literature of the last 8-9 years is rich of new theories and evidence against the predictive ability of Von Neumann Morgenstern model (see Loomes-Sudgen,1982,1987;Sudgen,1987;Machina,1987;Fishburn,1987;Sch·

oemaker,1982).

8.AN INDIRECT UTILITY INTERPRETATION

Now we want to see whether our model of decision making under uncertainty is able to give reason of the above choices which apparently contraddict the VM model.
Consider the indirect utility function which gives the maximum utility for given prices and income .As shown above we can reinterpret this function with probabilities as prices and the expected value of a prospect as the budget constraint.
In the case of three events and of a Cobb Douglas direct utility function, we can have

$$v(p,E) = (E / K\ p_1^{a} p_2^{b} p_3^{1-a-b}) \qquad (9)$$

where a+b<1 and K is a constant depending on a and b

We can eliminate the constant and take logs in (9) obtaining:

$$v(p,E) = \ln E - a \ln p_1 - b \ln p_2 - (1-a-b) \ln p_3 \qquad (10)$$

Now we show that for some values of the parameters of the equation (10) we can get the same pattern of preferences seen in the experiment conducted by Kahneman Tversky on the Allais paradox.
Let us consider a Cobb Douglas with the following parameters:
a=.2
b=.6
1-a-b=.2

Consider each prospect in turn and multiplying everything by 100 to avoid taking logs of numbers < 1 : A:

$E = 2500 \times 33 + 2400 \times 66 = 240900$

$v_A = \ln 240900 - .2 \ln 33 - .6 \ln 66 = 12.39 - .70 - 2.51 = 9.18$

B:

$E = 240000$

$v_B = \ln 240000 - .6 \ln 100 = 12.38 - 2.76 = 9.61$

Hence B P A.

Consider C and D:

C:

$E = 2500 \times 33 = 82500$

$v_C = \ln 82500 - .2 \ln 33 = 11.32 - .70 = 10.62$

D:

$E = 2400 \times 34 = 81600$

$v_D = \ln 81600 - .6 \ln 34 = 11.31 - 2.12 = 9.19$

hence C P D.

This is the same pattern of preferences displayed in Kahneman-Tversky,1979 experiment. Here it can be explained by simply using an indirect utility function of the kind proposed above.

9.SOME OBJECTIONS

Someone might object that it is a matter of the parameters of the Cobb Douglas chosen.To this objection the answer is immediate and is based on the consideration that prizes may have any kind of substitutability among them and hence any parameter may be justified.

A second objection may arise because of the use of the indirect utility function not just to compare maximum utilities yet in such guise,i.e. with probabilities instead of prices and a budget constraint (or expenditure) given by the expected value of a prospect.

Here the answer appears more complex.

First of all the problem of prices as probabilities.It appears that the consideration of a price as equivalent to the probability of an event might give rise to some criticism.However we said above that this price is a sort of shadow price of the player which has notnecessarily any relationship with the market price of the prospect.On the other hand we know that the price of a prospect can be well approximated by its expected value, making then probabilities more akin to some market price of events or prizes.

Secondly we should investigate a little more deeplyinto the properties of an <u>indirect utility function</u> like the one used.

To do that we go back to the properties of a traditional indirect utility function to see whether they continue to hold or are in need of alterations required by the new task to which they are called for.

Let us list the properties of traditional iuf's (Varian,1984):

1.v(**p**,E) is continuos for all **p**>>0 and E>0

2.v(**p**,E) is nonincreasing in **p** and non decresing in E

3.v(**p**,E) is quasi convex in **p**;

4.v(**p**,E) is homogeous of degree 0 in **p** and E.

Which are the properties still making sense in the new decision environment ? Which do not make any more sense ? Shall we still have an indirect utility function anyhow?

1.This one is still true in this version of the iuf even though we have to avoid cases in which some p are zero because we loose continuity, which is not a plight in itself but it makes the derivation of optimal bundles of consequences quite akward.
In all other cases the continuity is assured by the application of the maximum theorem (see Varian,1984 pg.121 and pg.326).

2.The first part sounds a little unfair if referred to probabilities rather than prices.However most of the times in consumer theory a normalization of relative prices is used making them vary between 0 and 1.The fact is that here the sum of prices=probabilities has to equal 1 or 10 or 100 (according to the manner we define probabilities). Saying that a vector **p** is greater than a vector **p**′ does not mean anything to us if $\Sigma_i p_i$ = 1 or 10 or 100 , since the definition of probability is unique despite the scale.There is no question about homogeneity in probabilities,unless we were to consider prospects which are not exhaustive or prospects giving other prospects as results.Only in those cases we can apply a proof based on preference sets (Varian,1984 pg.121).The fact that we cannot prove this property for our peculiar function does not have to detain us much since this property would not be very useful. Hence we can drop it without compliance.

3.As far as the third point is concerned it seems that we can retain the quasi convexity of v(**p**,E) in **p** ;let us try to see what it means and whether it is correct to retain quasi convexity.

Quasi convexity implies that

$(\mathbf{p}: v(\mathbf{p},E) \leq k)$ (11)

is a convex set for all $k \in R$. Even in this case we follow with few alterations the proof of Varian (1984). Suppose we have two vectors of probabilities **p** and **p**′ both satisfying (11) e consider a convex combination of the two vectors of the type

$t\mathbf{p}+(1-t)\mathbf{p}' = \mathbf{p}''$

we shall have that the iuf in which **p"** is substituted for **p** is still like (11). All **p** must satisfy the budget constraint hence we must have that

$\mathbf{p}\ \mathbf{x} \leq E$ (12)

$\mathbf{p}'\mathbf{x} \leq E$ (13)

$\mathbf{p}''\mathbf{x} \leq E$ (14)

If **x** is in (12) or in (13) it will also be in (14). If it was not so it would be

$t\mathbf{p}\mathbf{x} + (1-t)\mathbf{p}'\mathbf{x} > E$ (15)

but $\mathbf{p}\mathbf{x} \leq E$ and $\mathbf{p}\mathbf{x}' \leq E$.

The two latter inequality imply that

$t\mathbf{p}\mathbf{x} < tE$

$(1-t)\mathbf{p}'\mathbf{x} < (1-t)E$

which can be summed to give

$t\mathbf{p}\mathbf{x} + (1-t)\mathbf{p}'x < E$

which goes counter (15).

Hence $v(\mathbf{p}'',E) \leq \max u(\mathbf{x})$ because the set represented by (14) is contained in the union of (13) and (12).

4. This property is fairly irrelevant to us even though in some circumstances it could betreated as a sort of independence

property.Its proof is trivial.

10.CONCLUSIONS

In our presentation of a new approach to decision under risk we have emphasized two main aspects.

The first concerns the need to take into account in economic situations the conditions under which the supply of bets takes place.Supply of bets has to be considered because the availability of fair bets hinges upon the existence of certain kinds of suppliers.

The second is more enticing since it provides a sort of new criterion of decision under risk.This is based on the use of a utility function in which the arguments are the consequnces of an action and the probabilities are the price of any consequence.

Direct and indirect utility functions can then be used for choice under risk.Except for continuity the indirect utility function resembles very much the one under certainty.What is fairly surprising is that such an indirect utility function is able to explain the paradox of Allais without recuuring to non transitive decision criteria as it happens in most of recent literature.

In addition this approach seems very promising because the iuf can be used in a "near rationality " model (Akerlof-Yellen,1985) by the maximum theorem.

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