Advertising in a Di¤erential Game of Spatial Competition¤

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Abstract

We investigate a dynamic duopoly game with horizontal product di¤erentiation, to show that the standard approach to spatial competition fails to produce a pure strategy equilibrium in prices when treated in a di¤erential game framework. This holds independently of the shape of the transportation cost function. Then, we introduce an endogenous costs associated with the choice of location and characterise the open-loop and closed-loop equilibria of the model, showing that in the closed-loop case …rms invest more in product di¤erentiation and less in advertising, than they do in the open-loop setting. This happens because the gains from product di¤erentiation can be more easily internalised than those associated with advertising.

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1 Introduction

We propose a dynamic approach to the strategic use of non-price tools in a differential game model of spatial competition. Non-price variables typically include product and/or process R&D, product differentiation and advertising, that firms may use in isolation or together, so as to increase the profitability of their price or quantity strategies.[1] Here, we build upon Piga (1998), to focus on (i) horizontal differentiation, and (ii) advertising investments aimed at increasing demand (or market size).

Ever since Hotelling's (1929) seminal contribution, the role of product differentiation as a remedy to the fragility of market equilibrium under price competition has represented a core issue in the field of industrial organization.

However, under horizontal product differentiation, an established result is that a pure-strategy equilibrium in prices may not always exist.[2] More precisely, a subgame perfect equilibrium with prices greater than marginal cost may fail to exist, because firms' location choices drive prices to marginal cost when transportation costs are linear (or not sufficiently convex) in the distance between the generic consumer and the firm he decides to patronise. This non-existence problem has generated a stream of literature proposing several remedies, either by adopting non-linear transportation cost functions (d'Aspremont et al., 1979; Stahl, 1982; Economides, 1986) or by adopting the Stackelberg equilibrium as the solution concept (Anderson, 1987), or by choosing appropriate distribution functions for the population of consumers (de Palma et al., 1985; Neven, 1986), or a mix thereof (Tabuchi and Thisse, 1995; Lambertini, 1997a, 2000).

These remedies work in 'location-then-price' games, i.e., if the game is solved by backward induction with different variables being set at different stages. Novshek (1980) establishes that, if firms choose prices and locations simultaneously, then a pure strategy Nash equilibrium fails to exist due to an undercutting argument. This holds independently of consumer distributions and transportation cost functions, the only condition being that marginal costs must not be too steep. However, the backward induction algorithm widely used in static multistage games of product differentiation cannot be used to solve the continuous-time differential game formulations of the same problems.

[1] For a wide survey of these topics, see Tirole (1988) and Martin (1993).

[2] For exhaustive accounts of the debate, see Caplin and Nalebuff (1991); Anderson et al. (1992); Anderson et al. (1997).

We focus on this problem using as a benchmark a differential game model of advertising and horizontal product differentiation that can be found in Piga (1998). Transportation costs are linear as in Hotelling (1929), and firms' advertising investments increase the size of the market. That is, advertising is modelled as a public good. First we characterise the non-existence problem, and then we modify the setup to allow for a cost associated with the choice of locations. We establish the necessary and sufficient conditions ensuring the existence of a price equilibrium in pure strategies and we fully characterise the steady state equilibrium of the system. To this aim, we first adopt the open-loop equilibrium as the solution concept, whose outcome is then evaluated against the closed-loop equilibrium. In the latter case, which describes the strongly time consistent game, we establish that the only feedback in operation works through the choice of locations so as to induce firms to invest more in product differentiation and less in advertising, than they do in the open-loop setting. This is due to the fact that the gains from product differentiation can be more easily internalised than those associated with advertising.

The remainder of the paper is structured as follows. Section 2 illustrates the basic setup. The non-existence issue is investigated in section 3. Section 4 is devoted to the analysis of the model with costly location choice. Concluding remarks are in section 5.

2 The setup

We consider a market for horizontally differentiated products à la Hotelling (1929). Let the market exist over $t \in [0; \infty)$: Two profit-maximising firms, labelled as 1 and 2, choose locations $x_1(t)$ and $x_2(t) \in [0; 1]$ and compete in prices simultaneously as soon as both are in the market. Unit production cost is assumed to be constant and equal to c_i; $i = 1; 2$. Throughout the time horizon considered, both firms have the same discount rate $\rho \in [0; 1]$:

Consumers are uniformly distributed with density $N(t)$ along the unit interval $[0; 1]$: At any t; the total mass of consumers is therefore $N(t)$: The generic consumer located at $m \in [x_1; x_2]$ buys one unit of the good, enjoying the following net surplus:

$$U = s - p_i(t) - g(x_i(t) - m) \geq 0; \qquad i = 1; 2; \tag{1}$$

where x_i and p_i are firm's i location and mill price, respectively; $g(x_i - m)$ is the transportation cost function. In the remainder of the paper, we suppose that the reservation price s is never binding, so that full market coverage always obtains. If

$$g(x_i - m) \equiv k|x_i - m| \,; \tag{2}$$

the model keeps Hotelling's original assumption of linear disutility of transportation. Therefore, the consumer indifferent between products 1 and 2 is located at:[3]

$$\overline{m}(t) = \frac{p_2(t) - p_1(t) + k\,(x_1(t) + x_2(t))}{2k} \,; \tag{3}$$

and the associated demands are:

$$y_1(t) = N(t)\overline{m}(t) = \frac{N(t)\,[p_2(t) - p_1(t) + k\,(x_1(t) + x_2(t))]}{2k} \,; \; y_2(t) = N(t) - y_1(t) \,. \tag{4}$$

Otherwise, if

$$g(x_i - m) \equiv k(x_i - m)^2 \,; \tag{5}$$

the indifferent consumer locates at:

$$\tilde{m}(t) = \frac{[p_2(t) - p_1(t) + k\,(x_2^2(t) - x_1^2(t))]}{2k\,(x_2(t) - x_1(t))} \,; \tag{6}$$

and demand functions are defined as in d'Aspremont et al. (1979):

$$y_1(t) = N(t)\tilde{m}(t) = \frac{N(t)\,[p_2(t) - p_1(t) + k\,(x_2^2(t) - x_1^2(t))]}{2k\,(x_2(t) - x_1(t))} \,; \; y_2(t) = N(t) - y_1(t) \,. \tag{7}$$

Firms can increase the level of demand over time through the following dynamic equation:

$$\dot{N}(t) \equiv \frac{dN(t)}{dt} = \alpha\,[A_1(t) + A_2(t)] - \delta N(t) \,; \; \alpha > 0 \,; \tag{8}$$

where $A_i(t)$ is the advertising effort carried out by firm i at time t; and $\delta \in [0; 1]$ is the constant decay rate of demand. This type of advertising is a pure public good in the sense that the effort carried out by any firm benefits all firms alike (see Fershtman, 1984; Fershtman and Nitzan, 1991); accordingly, it is sometimes referred to as cooperative, with the implicit caveat

[3] Here, as well as in the case of quadratic disutility of transportation, we omit the indifference condition as well as the derivation of the expression for $\overline{m}(t)$; as they are well known from previous literature (see d'Aspremont et al., 1979, inter alia).

that firms do not cooperate in the sense of joint profit maximisation.[4] The instantaneous cost of advertising for firm i is:[5]

$$C_i(A_i(t)) = b[A_i(t)]^2 \; ; \; b > 0 \, . \tag{9}$$

Hence, firm i's instantaneous profits are:

$$\pi_i(t) = [p_i(t) - c_i]\, y_i(t) - b[A_i(t)]^2 \; , \tag{10}$$

where $y_i(t)$ is given, alternatively, by (4) or (7). Firm i's Hamiltonian is:

$$H_i(t) = e^{-\rho t} \cdot \Big\{ [p_i(t) - c_i]\, y_i(t) - b[A_i(t)]^2 + \lambda_i(t)\,[\alpha(A_1(t) + A_2(t)) - \delta N(t)] \Big\} \; , \tag{11}$$

where the control variables are $\{p_i(t); x_i(t); A_i(t)\}$; the state variable (common to both firms) is $N(t)$; and $\lambda_i(t) = \mu_i(t)e^{\rho t}$; $\mu_i(t)$ being the co-state variable associated to $N(t)$.

Two equilibrium concepts can be considered: the open-loop equilibrium and the closed-loop equilibrium. In general, these solutions do not coincide, the closed-loop equilibrium being subgame perfect while the open-loop equilibrium is not. However, there exist classes of differential games where the open-loop equilibrium is a degenerate closed-loop equilibrium, and therefore the two solutions coincide. In such a case, the open-loop equilibrium, if it exists, is also subgame perfect.[6] This feature characterises the present model, irrespective of whether the transportation cost function is linear or convex. To see this, it suffices to examine the first order (necessary) condition for the closed-loop equilibrium, associated to the co-state variable:[7]

$$-\frac{\partial H_i(t)}{\partial N(t)} - \frac{\partial H_i(t)}{\partial p_j(t)}\frac{\partial p_j^{br}(t)}{\partial N(t)} - \frac{\partial H_i(t)}{\partial x_j(t)}\frac{\partial x_j^{br}(t)}{\partial N(t)} = \frac{\partial \mu_i(t)}{\partial t} = \frac{\partial \lambda_i(t)}{\partial t} - \rho\lambda_i(t) \; ; \tag{12}$$

where superscript br stands for best reply, and the partial derivatives $\frac{\partial u_j^{br}(t)}{\partial N(t)}$; $u_j(t) = p_j(t); x_j(t)$; can be calculated on the basis of the best reply functions

[4] This labelling dates back to Friedman (1983). For a model where advertising is both cooperative and predatory, see Piga (2000, pp. 517-21).

[5] According to the cost function in (9), the advertising activity exhibits decreasing returns to scale. On the empirical evidence supporting this assumption, see Feichtinger et al. (1994).

[6] See Reinganum (1982a); Mehlmann and Willing (1983); Fershtman (1987); Fershtman et al. (1992).

[7] For an exhaustive exposition of the solution methods, see Başar and Olsder (1982, 1995[2]), Mehlmann (1988), Kamien and Schwartz (1981, 1991[2]).

obtaining from first order conditions concerning firm j's controls:

$$\frac{\partial H_j(t)}{\partial p_j(t)} = \frac{\partial \pi_j(t)}{\partial p_j(t)} = 0; \tag{13}$$

$$\frac{\partial H_j(t)}{\partial x_j(t)} = \frac{\partial \pi_j(t)}{\partial x_j(t)} = 0. \tag{14}$$

Now observe that both (13) and (14) contain the state variable $N(t)$ only in multiplicative form. Hence, best reply functions $p_j^{br}(t) = f_j(p_i(t); x_i(t); x_j(t))$ and $x_j^{br}(t) = g_j(p_i(t); p_j(t); x_i(t))$ are independent of $N(t)$. This entails that

$$\frac{\partial p_j^{br}(t)}{\partial N(t)} = \frac{\partial x_j^{br}(t)}{\partial N(t)} = 0; \tag{15}$$

and therefore, if a pure strategy equilibrium does exist, the open-loop equilibrium is a degenerate closed-loop equilibrium. Piga (1998) shows the coincidence between the open-loop equilibrium and the feedback equilibrium obtained through Bellman's value function approach, with $x_1 = 0$ and $x_2 = 1$. Therefore, at least for these locations, the feedback equilibrium also coincides with the closed-loop one. To this regard, two remarks are in order. The first is that, in general, the feedback equilibrium is a closed-loop equilibrium, while the opposite is not true (see ch. 6 in Başar and Olsder, 1982, 1995[2], inter alia). The second remark is that existence (and if so, the coincidence) of the three equilibria defined according to different information structures obtains for fixed locations. Hence, there arises a further issue, namely, whether this property holds once we allow firms to choose locations.

The issue of existence of equilibria is investigated in the next section.

3 The non-existence problem revisited

Consider the well known static approach to the linear transportation cost version of the Hotelling model, where the system of demand functions is (4). In d'Aspremont et al. (1979), it is proved that the undercutting incentive destroys the price equilibrium in pure strategies for all locations within the second and third quartiles of the linear city. In their contribution, this non-existence problem is shown to exist under the assumption that marginal production cost is the same for both firms. In the present setting, marginal costs will, in general, differ across firms. This introduces a further difficulty with the existence of a duopoly equilibrium in locations and prices, in that there may exist configurations of the vectors of control variables $\{p_i(t); p_j(t); x_i(t); x_j(t)\}$ and cost parameters $\{c_i; c_j\}$ where the market is a monopoly at the candidate equilibrium prices, even without considering the undercutting incentive. That is, the emergence of monopoly can be simply due to the difference in efficiency levels as measured by marginal production costs, if such a difference is sufficiently large to drive the inefficient firm out of business.[8]

We are going to show that, within the differential game approach, this market cannot produce a pure-strategy equilibrium in prices, irrespective of the shape of the transportation cost function. To see this, it suffices to examine the following argument. As we know from Novshek (1980), if firms choose prices and locations simultaneously, then a pure strategy Nash equilibrium fails to exist. In particular, (i) there can exist no equilibrium with firms located at different points, because then a firm would profit by choosing a location close to (or the same as) the rival's and undercut her price; and (ii) there is no equilibrium with homogeneous products, either because of a standard Bertrand argument leading to marginal cost pricing with each firm being induced to relocate away, if marginal costs are the same across firms, or to monopolization if marginal costs are different. This holds for all consumer distributions and transportation cost functions, provided marginal costs are not sharply U-shaped.[9]

Now consider that the solution method for a differential game consists in taking the first order conditions w.r.t. all control variables simultaneously, and observe that, on the basis of (13) and (14), the differential game formulated above reproduces the same first order conditions w.r.t. locations and prices that characterise the static game analysed by Novshek (1980). This establishes that the present game has no equilibrium in pure strategies, in that its solution is quasi-static w.r.t. prices and locations.

The same argument applies to all other settings where the differential

[8] Detailed calculations are omitted as they are straightforward. They are available from the authors upon request.

[9] For an exhaustive discussion of the non-existence problem when prices and locations are chosen simultaneously, see Beath and Katsoulacos (1991, ch. 2) and Anderson et al. (1992, ch. 8). In specific settings, the equilibrium existence can be restored through the so called 'no mill price undercutting' Nash equilibrium concept (see Kohlberg and Novshek, 1982), which, however, seems somewhat ad hoc.

equation(s) describing the dynamics of the state variable(s) is (are) unaffected by prices and locations. This is the case, for example, if $N(t) = N$ and firms invest so as to decrease transportation costs through a technology generically defined as follows:

$$\frac{dk(t)}{dt} = h\left(k(t); \Phi_i(t); \Phi_j(t)\right) \; ; \; \frac{\partial h(\cdot)}{\partial \Phi_i(t)} < 0 \; ; \tag{16}$$

where $\Phi_i(t)$ is the instantaneous effort produced by firm i: Such an effort can be interpreted either as an investment in advertising, aimed at reducing the 'perceived' disutility of buying a product which is not the preferred one, or as an investment in R&D to ameliorate the transportation technology.[10] In this setting, the Hamiltonian of firm i would be, for example, the following:

$$H_i(t) = e^{-\rho t} \cdot \left\{ [p_i(t) - c_i]\, y_i(t) - b\,[\Phi_i(t)]^2 - \lambda_i(t)\left[\alpha\left(\Phi_i(t) + \Phi_j(t)\right) - \delta k(t)\right] \right\} ; \tag{17}$$

where

$$C_i\left(A_i(t)\right) = b\,[\Phi_i(t)]^2 \tag{18}$$

is the instantaneous cost associated with investment $\Phi_i(t)$; and

$$\frac{dk(t)}{dt} = h\left(k(t); \Phi_i(t); \Phi_j(t)\right) = -\alpha\left(\Phi_i(t) + \Phi_j(t)\right) + \delta k(t) \tag{19}$$

describes the kinematics of the transportation cost rate $k(t)$:

Of course, the same holds if one considers the kinematic equations (8) and (19) jointly.

The foregoing discussion can be summarised as follows:

Proposition 1 *In any differential game of spatial competition where (i) each firm's price and location do not affect the dynamics of the state variable(s), and (ii) location is costless, there exists no duopoly equilibrium in pure strategies independently of the shape of the transportation cost function.*

4 A differential game with costly locations

From Proposition 1, we know that a pure strategy equilibrium fails to exist when (i) there is no cost associated with the choice of location, and (ii)

[10] Dos Santos Ferreira and Thisse (1996) illustrate a static model where firms can choose different transportation technologies in order to combine vertical and horizontal differentiation.

locations don't play any role in the kinematics of the state variable(s). In order to reformulate the model in such a way that it produces a pure strategy equilibrium, in this section we propose the following modification to Piga's setup.

First, by the symmetry of the model, we assume that $x_1(t) \le 1/2$ and $x_2(t) \ge 1/2$. That is, firms can locate also outside the city boundaries (as in Tabuchi and Thisse, 1995; Lambertini, 1997a). Then, we assume that location is costly. In particular, firms 1 and 2 bear, respectively, the following location costs:

$$\gamma_1[x_1(t)] = \beta[\xi_1 - x_1(t)]^2 \, ; \; \gamma_2[x_2(t)] = \beta[\xi_2 - x_2(t)]^2 \; \beta > 0 . \tag{20}$$

Observe that both cost functions are convex in locations, with $\gamma_1[x_1(t)]$ taking its minimum (equal to zero) at ξ_1 and $\gamma_2[x_2(t)]$ taking its minimum at ξ_2. Note that the cost functions associated to the choice of locations need not be convex in order to ensure the existence of equilibrium.[11]

Transportation costs are linear, so that demand functions are given by (4). For the sake of simplicity, we assume that marginal cost is equal to c for both firms. The kinematic equation of $N(t)$ is given by (8).Therefore, the relevant Hamiltonians are:

$$\begin{aligned} H_1(t) &= e^{-\rho t} \cdot \Big\{ [p_1(t) - c]\, y_1(t) - \beta[\xi_1 - x_1(t)]^2 - b[A_1(t)]^2 + \\ &\quad + \lambda_1(t)\, [\alpha(A_1(t) + A_2(t)) - \delta N(t)] \Big\} \, ; \end{aligned} \tag{21}$$

$$\begin{aligned} H_2(t) &= e^{-\rho t} \cdot \Big\{ [p_1(t) - c]\, y_1(t) - \beta[\xi_2 - x_2(t)]^2 - b[A_2(t)]^2 + \\ &\quad + \lambda_2(t)\, [\alpha(A_1(t) + A_2(t)) - \delta N(t)] \Big\} \, . \end{aligned} \tag{22}$$

4.1 The open-loop equilibrium

The first order conditions for the open-loop solution are:[12]

$$\frac{\partial H_1(t)}{\partial p_1(t)} = \frac{N(t)}{2k}\, [p_2(t) - 2p_1(t) + c + k\,(x_1(t) + x_2(t))] = 0 \, ; \tag{23}$$

[11] For a discussion, see Lambertini (1997b), where analogous cost functions are used to model optimal taxation in the static version of the Hotelling model with quadratic transportation costs.

[12] Second order conditions are satisfied here as well as in the calculations performed in the closed-loop case. They are omitted for the sake of brevity. Observe that, given the changes we have introduced in the model, there is now no presumption that either the feedback solution may coincide with either the open-loop or the closed-loop equilibria.

$$\frac{\partial H_2(t)}{\partial p_2(t)} = \frac{N(t)}{2k}\left[p_1(t) - 2p_2(t) + c + k(2 - x_1(t) - x_2(t))\right] = 0, \tag{24}$$

$$\frac{\partial H_1(t)}{\partial x_1(t)} = \frac{4\beta\left[\theta_1 - x_1(t)\right] + N(t)\left[p_1(t) - c\right]}{2} = 0, \tag{25}$$

$$\frac{\partial H_2(t)}{\partial x_2(t)} = \frac{4\beta\left[\theta_2 - x_2(t)\right] - N(t)\left[p_2(t) - c\right]}{2} = 0, \tag{26}$$

$$\frac{\partial H_i(t)}{\partial A_i(t)} = -2bA_i + \lambda_i\alpha = 0,\ i = 1, 2, \tag{27}$$

$$-\frac{\partial H_1(t)}{\partial N(t)} = \frac{\partial \lambda_1(t)}{\partial t} - \rho\lambda_1(t) \Rightarrow \tag{28}$$

$$\frac{\partial \lambda_1(t)}{\partial t} = (\rho + \delta)\lambda_1(t) - \frac{\left[p_1(t) - c\right]\left[p_2(t) - p_1(t) + k(x_1(t) + x_2(t))\right]}{2k},$$

$$-\frac{\partial H_2(t)}{\partial N(t)} = \frac{\partial \lambda_2(t)}{\partial t} - \rho\lambda_2(t) \Rightarrow \tag{29}$$

$$\frac{\partial \lambda_2(t)}{\partial t} = (\rho + \delta)\lambda_2(t) - \frac{\left[p_2(t) - c\right]\left[p_2(t) - p_1(t) + k(2 - x_1(t) - x_2(t))\right]}{2k}.$$

Moreover, we have the initial condition $N(0) = N_0$, and the transversality conditions which are omitted for the sake of brevity.

Henceforth, we drop the indication of time for the ease of exposition. Now we can solve the game, starting with the FOCs with respect to prices. From (23-24), we obtain:

$$p_1^* = \frac{3c + k(2 + x_1 + x_2)}{3},\ p_2^* = \frac{3c + k(4 - x_1 - x_2)}{3} \tag{30}$$

which can be plugged into the FOCs w.r.t. locations. Conditions (25-26) yield:

$$\begin{aligned} x_1^* &= \frac{2\beta kN(\theta_1 - \theta_2 - 2) - 24\beta^2\theta_1 + k^2N^2}{4\beta(kN - 6\beta)}, \\ x_2^* &= -\frac{2\beta kN(\theta_1 - \theta_2 - 4) - 24\beta^2\theta_2 + k^2N^2}{4\beta(kN - 6\beta)}. \end{aligned} \tag{31}$$

Candidate equilibrium prices (30) rewrite as follows:

$$\begin{aligned} p_1^* &= \frac{2\beta k(2 + \theta_1 + \theta_2) - kN(c + k) - 6c\beta}{6\beta - kN}, \\ p_1^* &= \frac{2\beta k(4 - \theta_1 - \theta_2) - kN(c + k) - 6c\beta}{6\beta - kN}. \end{aligned} \tag{32}$$

Necessary and sufficient conditions for a pure strategy price equilibrium to exist are:

$$\theta_1 < \frac{\beta - kN}{4\beta}\,;\ \theta_2 > \frac{3\beta + kN}{4\beta} \tag{33}$$

which ensure that $x_1^* < 1/4$ and $x_2^* > 3/4$. Then, from (27) we have that

$$\lambda_i = \frac{2bA_i}{\alpha}\,; \tag{34}$$

and

$$\frac{\partial A_i}{\partial t} = \frac{\alpha}{2b} \cdot \frac{\partial \lambda_i}{\partial t}\,. \tag{35}$$

Using (28), (29), (31) and (35), we obtain the following expressions:

$$A_1^* = \frac{\alpha k\left[2\beta(2+\theta_1+\theta_2) - kN\right]^2}{4b(kN-6\beta)^2(\rho+\delta)}\,;\qquad A_1^* = \frac{\alpha k\left[2\beta(4-\theta_1-\theta_2) - kN\right]^2}{4b(kN-6\beta)^2(\rho+\delta)}\,; \tag{36}$$

which can be further simplified by invoking the symmetry condition $\theta_2 = 1 - \theta_1$, yielding:

$$A_1^* = A_2^* = A^* = \frac{\alpha k}{4b(\rho+\delta)}\,. \tag{37}$$

This imposition can be justified on the following grounds. Parameter θ_i represents the location at which firm i's relocation costs are zero. Given the a priori symmetry of the model as to all other features, it appears reasonable to assume also that cost-minimising locations are symmetric around 1/2. Obviously, his also entails $x_2^* = 1 - x_1^*$. Steady state equilibrium prices are $p_i^* = c + k$.

As a last step, from (8), we obtain the steady state value of N:

$$N^* = \frac{\alpha(A_1^* + A_2^*)}{\delta}\,;\ \text{under } \theta_2 = 1 - \theta_1;\ N^* = \frac{2\alpha A^*}{\delta} = \frac{\alpha^2 k}{2b\delta(\rho+\delta)}\,. \tag{38}$$

The discussion carried out so far can be summarised as follows:

Proposition 2 If the following conditions hold:

$$c_i = c\,;\ \theta_1 < \frac{\beta - kN}{4\beta}\,;\ \theta_2 = 1 - \theta_1 > \frac{3\beta + kN}{4\beta}\,;$$

$$\beta \geq \frac{\alpha^2 k^2}{4b\delta(4\rho+3\delta)}\,;$$

then, the open-loop differential game of advertising with costly location choice admits a unique steady state equilibrium where:

$$p_1^{OL} = p_2^{OL} = c + k\,;$$

$$x_1^{OL} = \eta_1 + \frac{kN^{OL}}{4\beta} = \eta_1 + \frac{\alpha^2 k^2}{8b\beta\delta(\rho+\delta)}\,;$$

$$x_2^{OL} = 1 - x_1^{OL} = 1 - \eta_1 - \frac{kN^{OL}}{4\beta} = 1 - \eta_1 - \frac{\alpha^2 k^2}{8b\beta\delta(\rho+\delta)}\,;$$

$$A_i^{OL} = \frac{\alpha k}{4b(\rho+\delta)}\,;$$

$$N^{OL} = \frac{\alpha\left(A_i^{OL} + A_j^{OL}\right)}{\delta} = \frac{\alpha^2 k}{2b\delta(\rho+\delta)}\,;$$

$$\pi_i^{OL} = \frac{\alpha^2 k^2\left[4b\beta\delta(4\rho+3\delta) - \alpha^2 k^2\right]}{64 b^2\beta\delta^2(\rho+\delta)^2}\,.$$

As to the stability of the dynamic system, the following holds:

Proposition 3 The steady state

$$A_i^{OL} = \frac{\alpha k}{4b(\rho+\delta)}\,;$$

$$N^{OL} = \frac{\alpha\left(A_i^{OL} + A_j^{OL}\right)}{\delta} = \frac{\alpha^2 k}{2b\delta(\rho+\delta)}\,.$$

is a saddle point.

Proof. See the appendix. ■

It is worth noting that, irrespective of firms' locations, the symmetry conditions (a) $\eta_2 = 1 - \eta_1$ and (b) $c_1 = c_2 = c$ suffice to yield $p_i^{OL} = c + k$, which is the same price as in Piga (1998, Proposition 3.1, p. 513) under (b).

Now adopt condition (a), and examine:[13]

$$\frac{\partial\left(x_2^{OL} - x_1^{OL}\right)}{\partial\delta} = \frac{\alpha^2 k^2(\rho+2\delta)}{4b\beta\delta^2(\rho+\delta)^2} > 0\,; \qquad (39)$$

[13] The properties (39) and (40) hold in general. They are derived under assumption (a) for simplicity.

and
$$\frac{\partial\left(x_2^{OL}-x_1^{OL}\right)}{\partial\rho}=\frac{\alpha^2k^2}{4\beta\delta\left(\rho+\delta\right)^2}>0\,. \tag{40}$$

Moreover, from (36) or (37), it is immediately clear that $\partial A_i^{OL}/\partial\delta$ and $\partial A_i^{OL}/\partial\rho$ are both negative. Therefore, we can state:

Corollary 1 Under the open-loop solution, the steady state degree of differentiation increases both in the discount rate and in the decay rate. The opposite holds for the optimal investment in advertising.

The above results can be reformulated in the following terms. As the decay rate and discounting increase, the incentive for firms to advertise in order to sustain demand becomes weaker, and they use a larger differentiation as an alternative instrument to increase their profitability. In a sense, larger values of both δ and ρ tend to shorten the perceived duration of the game, and therefore firms find it convenient to exploit product differentiation in a quasi-static fashion, rather than focussing upon the intertemporal demand increase through advertising.

4.2 The closed-loop equilibrium

On the basis of Hamiltonians (21) and (22), it can be immediately established that the first order conditions on controls are as in (23-27). The relevant differences appear in the co-state equations, which are now defined as follows:

$$-\frac{\partial H_i(t)}{\partial N(t)}-\frac{\partial H_i(t)}{\partial p_j(t)}\frac{\partial p_j^{br}(t)}{\partial N(t)}-\frac{\partial H_i(t)}{\partial x_j(t)}\frac{\partial x_j^{br}(t)}{\partial N(t)}-\frac{\partial H_i(t)}{\partial A_j(t)}\frac{\partial A_j^{br}(t)}{\partial N(t)}=\frac{\partial\lambda_i(t)}{\partial t}-\rho\lambda_i(t)\,. \tag{41}$$

Examine the game from firm 1's standpoint. From firm 2's first order conditions on control variables, we have:

$$\frac{\partial p_2^{br}(t)}{\partial N(t)}=\frac{\partial A_2^{br}(t)}{\partial N(t)}=0\,; \tag{42}$$

while
$$\frac{\partial x_2^{br}(t)}{\partial N(t)}=\frac{c-p_2(t)}{4\beta}\,. \tag{43}$$

Using (43) and

$$\frac{\partial H_1(t)}{\partial x_2(t)} = \frac{N(t)\,[p_1(t) - c]}{2}\,; \tag{44}$$

we obtain the co-state equation pertaining to firm 1's closed-loop problem:

$$\begin{aligned}\frac{\partial \lambda_1(t)}{\partial t} &= (\rho + \delta)\,\lambda_1(t) - \frac{[p_1(t) - c]\,[p_2(t) - p_1(t) + k\,(x_1(t) + x_2(t))]}{2k} + \\ &\quad + \frac{N(t)\,[p_1(t) - c]\,[p_2(t) - c]}{8\beta}\,. \end{aligned} \tag{45}$$

Now, following the same procedure as in the open-loop case (in particular, using (30-35) and $\eta_2 = 1 - \eta_1$), we obtain the results summarised in the following Proposition:

Proposition 4 Given $\eta_2 = 1 - \eta_1$, then, the closed-loop differential game of advertising with costly location choice admits a unique steady state equilibrium where:

$$p_i^{CL} = c + k\,;$$

$$x_1^{CL} = \frac{8b\beta\delta\eta_1(\rho + \delta) + \alpha^2k^2(1 + \eta_1)}{8b\beta\delta(\rho + \delta) + \alpha^2k^2}\,;\; x_2^{CL} = 1 - x_1^{CL}\,;$$

$$A_i^{CL} = \frac{2\alpha\beta\delta k}{8b\beta\delta(\rho + \delta) + \alpha^2k^2}\,;\; N^{CL} = \frac{2\alpha A_i^{CL}}{\delta} = \frac{4\alpha^2\beta k}{8b\beta\delta(\rho + \delta) + \alpha^2k^2}\,;$$

$$\pi_i^{CL} = \frac{\alpha^2k^2\left[256b^3\beta^3\delta^3(\rho + \delta)^2(4\rho + 3\delta) + \alpha^2k^2\,\Omega\right]}{64b^2\beta^2\delta^2(\rho + \delta)^2\left[\alpha^2k^2 + 8b\beta\delta(\rho + \delta)\right]^2}\,;$$

where

$$\Omega = 64\,[b\beta\delta(\rho + \delta)]^2 - \alpha^2k^2\left[\alpha^2k^2 + 16b\beta\delta(\rho + \delta)\right]\,.$$

Appropriate conditions on parameters can be established to ensure the non-negativity of the closed-loop equilibrium profits π_i^{CL}. In particular, focus on the (intertemporal) marginal productivity of advertising, α, and define $\alpha^2 \equiv \zeta$. The equation $\pi_i^{CL} = 0$ has four roots in ζ, out of which two are not real and the remaining two are $\zeta_1 = 0$ and $\zeta_2 > 0$. Then, $\pi_i^{CL} > 0$ for all $\zeta \in (0, \zeta_2)$, while $\pi_i^{CL} < 0$ for all $\zeta > \zeta_2$.

The stability analysis of $\left\{A^{CL}, N^{CL}\right\}$ produces the following:

Proposition 5 The pair $\left\{A^{CL}, N^{CL}\right\}$ is a saddle point.

Proof. See the appendix. ■

Proposition 4 has the following Corollary:

Corollary 2 Under the closed-loop solution, the steady state degree of differentiation increases both in the discount rate and in the decay rate. The optimal investment in advertising is (i) always decreasing in the discount rate, while (ii) it is increasing in the decay rate iff

$$\delta \in \left[0, \min\left\{\frac{\alpha k\sqrt{2b\beta}}{4b\beta}, 1\right\}\right)$$

and conversely.

Proof. Proving the effect of parameters δ and ρ on the steady state degree of differentiation is straightforward, by using x_i^{CL}. The same applies to the effect on A_i^{CL} of a change in ρ. Point (ii) is proved by:

$$\frac{\partial A_i^{CL}}{\partial \delta} \propto \alpha^2 k^2 - 8b\beta\delta^2. \tag{46}$$

■

Again, we use $\alpha^2 \equiv \gamma$ to verify that the value of γ at which $\partial A_i^{CL}/\partial \delta = 0$, i.e., $\gamma_3 = 8b\beta\delta^2/k^2$, belongs to the interval $(0, \gamma_2)$. As an example, if we set $b = \beta = 1/2$, $k = 1$ and $\delta = \rho = 1/10$, we obtain $\gamma_2 \simeq 0.0383 > \gamma_3 = 1/50$.

4.3 A comparison of open-loop and closed-loop equilibria

We are now in a position to carry out a comparative assessment of the closed-loop equilibrium against the open-loop one. Under $\eta_2 = 1 - \eta_1$, it is a matter of straightforward calculations to establish that:

$$A_i^{CL} < A_i^{OL} \Rightarrow N^{CL} < N^{OL}. \tag{47}$$

Moreover, from (43) we know that the equilibrium value of x_2 increases as N decreases, and conversely. This also entails that the optimal value of x_1 increases as N increases, since $x_2 = 1 - x_1$. Hence,

$$x_1^{CL} < x_1^{OL} \text{ and } x_2^{CL} > x_2^{OL} \tag{48}$$

which entails that products are more differentiated at the closed-loop equilibrium than at the open-loop one. This discussion leads to our final result:

Proposition 6 The comparison between the open-loop equilibrium and the closed-loop equilibrium reveals that (i) the equilibrium price is the same under both solution concepts; (ii) product differentiation is larger under the closed-loop equilibrium; (iii) adverting is more intense and the resulting demand level is higher under the open-loop equilibrium.

The established wisdom concerning investment behaviour in dynamic games maintains that firms invest less (in R&D or capacity) in closed-loop and feedback equilibria than in open-loop ones (see Reinganum, 1981; 1982b; Reynolds, 1987, inter alia). This model provides a counterexample related to investment in demand-increasing activities. This result can be interpreted on the following grounds. Notice that points (ii) and (iii) in the above Proposition can be attributed to the presence in the co-state equations of the feedback of the state variable through the location choice only. This amounts to saying that, in the subgame perfect equilibrium, firms prefer to invest more in product differentiation than in demand-increasing advertising, as compared to what they do in the open-loop equilibrium which is only weakly time consistent and therefore requires firms to commit themselves forever to the plan designed at the initial date. In this game, an increase in demand is a substitute for an increase in differentiation (and conversely) as both contribute to increase instantaneous revenues. Given the tradeoff highlighted by the closed-loop decision rule in (43), which, by definition, does not appear in the open-loop formulation, in a strongly time consistent equilibrium firms are lead to invest less in advertising and more in product differentiation than they would if they were to design their respective plans once and for all at $t = 0$. The reason can be found in the cooperative nature of advertising, i.e., in its being a public good, while the benefits from product differentiation can be more easily internalised.

In the closed-loop game, the rate of depreciation of demand positively affects the optimal investment in advertising, as long as the decay rate itself is below the critical threshold defined in Corollary 2. However, as we know from Proposition 6, this may only partially counterbalance the substitution operated in favour of product differentiation.

The final step consists in evaluating steady state profits in the two equilibria. To this purpose, we have the following:

$$\pi_i^{OL} - \pi_i^{CL} = \frac{\alpha^4 k^4 \left[16b\beta\delta\left(\delta^2 + 3\delta\rho + 2\rho^2\right) + \alpha^2 k^2 (4\rho + 3\delta)\right]}{16b\delta(\rho+\delta)^2\left[\alpha^2 k^2 + 8b\beta\delta(\rho+\delta)\right]^2} > 0 . \quad (49)$$

Therefore, the following holds:

Proposition 7 In steady state, firm's profits are higher in the open-loop equilibrium than in the closed-loop equilibrium.

The above Proposition reflects a rather common result in differential games, namely, that committing to a production and/or investment plan at the outset ensures higher profits compared to the subgame perfect equilibrium where each player is allowed to react optimally to rivals at any time along the path to the steady state (see Fershtman and Kamien, 1987; Reynolds, 1987; Mehlmann, 1988, ch. 5; Cellini and Lambertini, 2001). Although regularly stressed in the existing literature on dynamic games, this result has to be evaluated taking into account a caveat, namely, that firms may not be able to choose at all between the open-loop and the closed-loop solution and therefore such inequality describes a comparative statics property of the dynamic game but has no particular bearings as to firms' preferences on how to play it. In particular, if firms are to play in a strongly time consistent way, the open-loop is ruled out and any inequality on profits such as (49) is just irrelevant.

5 Concluding remarks

Taking the advertisement game illustrated in Piga (1998) as a starting point, we have proved that the standard approach to horizontal differentiation cannot produce a pure strategy equilibrium in prices when treated in a differential game framework. This is due to the same undercutting mechanism investigated by Novshek (1980). Moreover, this result holds true irrespective of the shape of the transportation cost function.

Then, we have introduced an endogenous costs associated with the choice of location. This has allowed us to characterise (i) the necessary and sufficient conditions for existence of a pure strategy equilibrium, and (ii) the steady state of the model, adopting alternatively the open-loop and the closed-loop solution concepts. We have shown that in the closed-loop case firms invest more in product differentiation and less in advertising, than they do in the open-loop setting. This happens because the gains from product differentiation can be more easily internalised than those associated with advertising.

Appendix

Proof of Proposition 3. First, observe that steady state prices and locations are quasi-static, in the sense that they can be calculated (in terms of $N(t)$) from first order conditions on Hamiltonians (21) and (22), without deriving their kinematic equations. Therefore, the stability analysis can be confined to the dynamics of $A_i(t)$ and $N(t)$, evaluated at $\left\{A^{OL}; N^{OL}\right\}$. The joint dynamics of A and N can be described by linearising (8) and (35) around $\left\{A^{OL}; N^{OL}\right\}$, to get what follows:

$$\begin{bmatrix} \dot{N} \\ \dot{A} \end{bmatrix} = \Xi \begin{bmatrix} (N - N^{OL}) \\ (A - A^{OL}) \end{bmatrix} \tag{50}$$

where

$$\Xi = \begin{bmatrix} -\delta & \alpha \\ 0 & \rho + \delta \end{bmatrix}$$

The stability properties of the system in the neighbourhood of the steady state depend upon the trace and determinant of the 2×2 Jacobian matrix Ξ. In studying the system, we confine to steady state points. The trace of Ξ is $tr(\Xi) = \rho > 0$, whereas the determinant $\Delta(\Xi) = -\delta(\rho + \delta) < 0$. Therefore, $\left\{A^{OL}; N^{OL}\right\}$ is a saddle point. ■

Proof of Proposition 5. The procedure is the same as in the proof of Proposition 3. The Jacobian matrix becomes:

$$\Xi = \begin{bmatrix} -\delta & \alpha \\ \frac{\alpha(p_1 - c)(p_2 - c)}{16b\beta} & \rho + \delta \end{bmatrix}$$

with $tr(\Xi) = \rho > 0$ and

$$\Delta(\Xi) = -\delta(\rho + \delta) - \frac{\alpha^2 k^2}{16b\beta} < 0.$$

Therefore, $\left\{A^{CL}; N^{CL}\right\}$ is a saddle point. ■

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