Analysis of an uncertain volatility model *

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Abstract

We examine, both from an analytical and numerical viewpoint, the uncertain volatility model by Hobson-Rogers in the framework of degenerate parabolic PDEs of Kolmogorov type.

1 Introduction

Several extensions of the Black-Scholes model [3] have appeared in literature (see, for a survey, Epps [5]) aiming to capture the characteristic observed patterns of the implied volatility given by the market. Here we are concerned with the seemingly promising model proposed by Hobson and Rogers [6] who assume that the volatility is a deterministic function of the history of the spot process. The merit of the model is twofold: first, it is potentially capable to reproduce smiles, skews of different directions and volatility terms structures. Second, it preserves the completeness of the market since no exogenous source of risk is added, so that the classical arbitrage pricing and hedging theory applies. In particular, there are unique preference-independent prices for claims given in terms of the expectation under an equivalent martingale measure or as solutions to a PDE in three variables. Indeed incorporating the dependence on past prices enters an additional state variable on which the derivative's price depends: then, as in the case of Asian or look-back options, the associated PDE is of degenerate parabolic type.

In this paper we focus on the analytical and numerical treatment of the Hobson-Rogers model in the framework of Kolmogorov PDEs. Degenerate equations of Kolmogorov type naturally arise in the problem of pricing path dependent contingent claims. The simplest significant example is given by Asian-style derivatives: if we assume that the stock price S_t is a standard geometric Brownian motion with volatility σ, then the price U of a geometric average Asian option is a solution to the equation

$$\partial_t U + rS\partial_S U + \frac{1}{2}\sigma^2 S^2 \partial_{SS} U + \log(S)\partial_A U = rU, \tag{1.1}$$

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where "A" denotes the path-dependent variable and r is the risk-free rate (see, for instance, [16] and [2]). By an elementary change of variables, (1.1) can be reduced to the following PDE in $\mathbb{R}^3$

$$\mathcal{K}u := \partial_{xx}u + x\partial_y u - \partial_t u = 0. \tag{1.2}$$

Although (1.2) is strongly degenerate due to the lack of diffusion in the y-direction, Kolmogorov [7] constructed an *explicit* fundamental solution to (1.2) of Gaussian type, which is a C^∞ function outside the diagonal (cf. (2.9)). Consequently equation (1.2) has a closed form solution and is hypoelliptic, that is every distributional solution to (1.2) is a C^∞ function.

In this paper we briefly introduce to the theory of Kolmogorov PDEs and describe their link with uncertain volatility models. Then we consider the numerical solution to the option pricing problem in the Hobson-Rogers model by finite-differences schemes: our main goal is to provide some new non-Euclidean schemes which seem to be particularly efficient if compared with the classical ones. In the last part of this note we present some empirical results.

2 Hobson-Rogers model and Kolmogorov equations

Hobson and Rogers propose in [6] a complete-market model with uncertain volatility. Fixed a maturity T and a probability space $(\Omega, \mathcal{F}, P)$ with one-dimensional Brownian motion (B_t), we denote by S_t the underlying price and by D_t the deviation of prices from the trend, defined by

$$D_t = Z_t - \int_0^{+\infty} \lambda e^{-\lambda\tau} Z_{t-\tau} d\tau, \qquad \lambda > 0, \tag{2.1}$$

where $Z_t = \log(e^{-rt}S_t)$ is the discounted log-price. In (2.1), the parameter λ amounts to the rate at which past prices are discounted. Hobson and Rogers assume that S_t is an Ito process satisfying

$$dS_t = \mu(D_t)S_t dt + \sigma(D_t)S_t dB_t, \tag{2.2}$$

where μ and σ are deterministic functions and σ is positive. Existence and uniqueness of the solution (S_t, D_t) to the system of SDEs (2.2)-(2.1) are guaranteed by the usual Lipschitz assumptions on μ and σ.

The main advantage of the Hobson-Rogers model is that no exogenous source of risk has been included. Then the complete market setting is preserved and usual no-arbitrage arguments provide unique option prices. In particular the derivative's price can be represented as the conditional expectation of the payoff under the (unique) P-equivalent risk-neutral measure. On the other hand, the deviation (2.1) enters as an additional state variable on which the option price depends. Then the drawback is that the augmented PDE associated to the model is of degenerate type. Indeed, let

$$U_{T-t} = f(S_t, D_t, t)$$

denote the price of a contingent claim at time $T - t$. By the Feynman-Kac formula the function f satisfies the PDE in $\mathbb{R}^3$:

$$\frac{\sigma^2(D)}{2}(S^2\partial_{SS}f + \partial_{DD}f + 2S\partial_{DS}f - \partial_D f) + rS\partial_S f - \lambda D\partial_D f - \partial_t f = rf. \tag{2.3}$$

In the case of an European Put option with strike K, we also have the condition

$$f(S, D, 0) = (K - S)^+. \tag{2.4}$$

Equation (2.3) is of degenerate type since the quadratic form associated to its second order part is represented by the singular matrix

$$\frac{\sigma^2}{2} \begin{pmatrix} S^2 & S \\ S & 1 \end{pmatrix}.$$

Nevertheless Hobson and Rogers remark that, under the further hypothesis that σ is a smooth (C^∞) function, then Hörmander's Theorem on hypoelliptic PDEs applies and problem (2.3)-(2.4) has a classical solution.

Here we make the additional remark that (2.3) belongs to the noteworthy subclass of Hörmander PDEs today called of Kolmogorov or Ornstein-Uhlenbeck type. For this class a very satisfactory theory has been developed and many sharp results are available even under few regularity assumptions (see [8] for an exhaustive survey on this topic). We aim to show that the theory of Kolmogorov PDEs provides the natural framework for the study of the Hobson-Rogers model both from an analytical and a numerical viewpoint.

For what follows, it is convenient to rewrite equation (2.3) in the following form:

$$a(\partial_{xx}u - \partial_x u) + x\partial_y u - \partial_\tau u = 0 \tag{2.5}$$

where $u = u(x, y, \tau)$ is determined by the transformation

$$f(S, D, t) = Ke^{-rt}u\left(\log\left(\frac{S}{K}\right) + rt, e^{-\lambda t}\left(\log\left(\frac{S}{K}\right) + rt - D\right), 1 - e^{-\lambda t}\right), \tag{2.6}$$

and we have set

$$a(x, y, \tau) = \frac{\sigma^2\left(x - \frac{y}{1-\tau}\right)}{2\lambda(1-\tau)}, \qquad \tau \in [0, 1 - e^{-\lambda T}]. \tag{2.7}$$

By this change of variables, problem (2.3)-(2.4) is equivalent to the forward Cauchy problem for (2.5) in the strip $\mathbb{R}^2 \times [0, 1 - e^{-\lambda T}]$ with initial condition

$$u(x, y, 0) = (1 - e^x)^+, \qquad (x, y) \in \mathbb{R}^2. \tag{2.8}$$

Transformation (2.6) seems to be more convenient than the one proposed by Hobson and Rogers (cf. Sec.4.2 in [6]). This will be apparent in the next section where we investigate the numerical approximation. From a qualitative viewpoint, we note that the derivative $\partial_x u$ in (2.5) does not affect the main properties of the equation since it is, roughly speaking, "dominated" by the second order part $\partial_{xx}u$. Then we may consider equation (2.5) as a perturbation of the Kolmogorov type equation (1.2) with *constant* diffusion term. We recall that Kolmogorov [7] constructed explicitly a fundamental solution to (1.2). Precisely, we have that

$$\Gamma_K(z; \zeta) = \frac{\sqrt{3}}{2\pi(t-\tau)^2} \exp\left(-\frac{(x-\xi)^2}{4(t-\tau)} - \frac{3}{(t-\tau)^3}\left(y - \eta - \frac{t-\tau}{2}(x+\xi)\right)^2\right), \tag{2.9}$$

for $t > \tau$ and $\Gamma_{\mathcal{K}}(z;\zeta) = 0$ for $t \le \tau$ is the fundamental solution to $\mathcal{K}$ with pole at $\zeta = (\xi, \eta, \tau)$ and evaluated in $z = (x, y, t)$. We also recall that operator $\mathcal{K}$ has the remarkable property of being invariant with respect to the non-Euclidean left translations in the law

$$(x, y, t) * (\xi, \eta, \tau) = (x + \xi, y + \eta - x\tau, t + \tau), \tag{2.10}$$

and homogeneous of degree two with respect to the dilations

$$\delta_s(x, y, t) = (sx, s^3 y, s^2 t), \qquad s > 0, \tag{2.11}$$

in the sense that

$$\mathcal{K}(u \circ \delta_s) = s^2 (\mathcal{K} u) \circ \delta_s.$$

Then it is natural to introduce in $\mathbb{R}^3$ the δ_s-homogeneous norm

$$\|(x, y, t)\| = \left(x^6 + y^2 + |t|^3\right)^{\frac{1}{6}},$$

and the following notion of $\mathcal{K}$-Hölder continuity. Let $\alpha \in]0, 1[$ and Q be an open subset of $\mathbb{R}^3$; we say that a function $u : Q \longrightarrow \mathbb{R}$ is $\mathcal{K}$-Hölder continuous of order α (in short, $u \in C^\alpha_{\mathcal{K}}(Q)$) if

$$|u|_{\alpha,Q} := \sup_Q |u| + \sup_{\substack{z,\zeta \in Q \\ z \neq \zeta}} \frac{|u(z) - u(\zeta)|}{\|\zeta^{-1} * z\|^\alpha}$$

is finite. We remark explicitly that

$$C^\alpha(Q) \subseteq C^\alpha_{\mathcal{K}}(Q) \subseteq C^{\frac{\alpha}{3}}(Q)$$

where $C^\alpha(Q)$ denotes the space of Hölder continuous functions in the usual sense (see, for instance, [9]). Moreover we say that $u \in C^{2+\alpha}_{\mathcal{K}}(Q)$ if $u \in C^\alpha_{\mathcal{K}}(Q)$ and

$$\sup_Q |\partial_x u| + |\partial_{xx} u|_{\alpha,Q} + |Y u|_{\alpha,Q} < \infty$$

where $Yu = x\partial_y u - \partial_t u$ is the first order part of equation (1.2).

Remark 2.1 *Roughly speaking, one should consider* ∂_x *and* Y *respectively as a first and second order derivative intrinsic to* $\mathcal{K}$*. These are the main directional derivatives of the degenerate equation* (1.2) *in the sense that they allow to recover the other lacking directions. For instance,* ∂_y *can be obtained as the commutator of* ∂_x *and* Y*:*

$$\partial_y = \partial_x Y - Y \partial_x,$$

therefore it should be considered as a third order derivative in the intrinsic sense.

In some way $\mathcal{K}$ plays for (2.5) a role analogous to that of constant-coefficients operators in the classical theory of elliptic or parabolic PDEs (actually, a constant coefficient operator is nothing more than a translation-invariant operator). Then many results can be extended from (1.2) to (2.5) by perturbation arguments. In particular the next theorem, proved in [13], ensures the existence of a unique classical solution to the Cauchy problem for (2.5) under minimal regularity assumptions.

Theorem 2.2 *Assume that $a \in C^{\alpha}_{\mathcal{K}}(\mathbb{R}^2 \times]0,T[)$ for some $\alpha \in]0,1[$, $T > 0$ and $a \geq c$ for some positive constant c. Then equation (2.5) has a fundamental solution Γ and for every bounded $\varphi \in C(\mathbb{R}^2)$, the function*

$$u(x,y,t) = \int_{\mathbb{R}^2} \Gamma(x,y,t;\xi,\eta,0)\varphi(\xi,\eta)d\xi d\eta \tag{2.12}$$

belongs to $C^{2+\alpha}_{\mathcal{K}}(\mathbb{R}^2 \times]0,T[)$ and is the unique bounded, classical solution to the Cauchy problem for (2.5) with initial condition $u(x,y,0) = \varphi(x,y)$.

We also recall that global estimates of Γ (and its derivatives) in terms of $\Gamma_{\mathcal{K}}$ are proved in [14] and [10]; then sharp estimates of the solution u in (2.12) and of its derivatives are available. In particular we recall the following Gaussian type estimate of Γ:

$$\Gamma(z;\zeta) \leq C\ \Gamma^{c}_{\mathcal{K}}(z;\zeta), \qquad \forall z,\zeta \in \mathbb{R}^3, \tag{2.13}$$

where C, c are a positive constants and $\Gamma^{c}_{\mathcal{K}}$ denotes the (explicitly known) fundamental solution to the Kolmogorov operator $c\partial_{xx}u + x\partial_y u - \partial_t u$.

The non-Euclidean differential-geometrical structure naturally associated to the Kolmogorov equation (1.2) also gives some insight for the numerical approximation of the solution. We recall that in the paper by Barucci, Polidoro and Vespri [2] the price of a geometric average Asian option is represented in terms of the solution to the Kolmogorov equation (1.2) and the non-Euclidean finite differences scheme proposed in [15] is used for the numerical solution. In the next section we adapt this approach to the Hobson-Rogers model and perform a comparison among different numerical methods.

We close this section recalling that nonlinear Kolmogorov equations have been considered for pricing options with memory feedback by Peszek [12] and for a stochastic differential utility problem by Antonelli and Pascucci [1], Pascucci and Polidoro [11].

3 Finite-difference schemes for the Hobson-Rogers model

We consider the numerical solution of the option pricing PDE (2.3) by finite-difference methods. By simplicity we assume $r = 0$. A standard approach consists in approximating the derivatives ∂_S, ∂_D and ∂_t by finite differences: we refer to this as an *Euclidean finite-difference scheme*. In view of Remark 2.1, it seems very natural to study the Kolmogorov equation (2.5) by approximating its main directional derivatives ∂_x and $Y = x\partial_x - \partial_t$ rather than the usual Euclidean derivatives: we call this a *Kolmogorov finite-difference scheme*. It turns out that this last scheme allows a better comprehension of the discrete structure of the equation and provides very efficient approximations: we refer to the next section for a comparison of numerical methods. We recall that Kolmogorov finite-difference schemes for (1.2) were first proposed by Mogavero and Polidoro in [15] and applied to the Hobson-Rogers model by Di Francesco and Pascucci in [4] .

Solving options pricing PDEs by finite-difference methods requires the discretization of the equation in a bounded region. Therefore a primary question arises about the choice of the initial-boundary conditions. This specification seems to be unavoidable in the case of an Euclidean finite-difference scheme and it is usually made relying upon financial considerations.

On the contrary, as we shall see in Subsection 3.1, it is possible to derive boundary conditions for Kolmogorov schemes in bounded domains by purely analytical considerations. This is the first apparent advantage of Kolmogorov on Euclidean schemes.

3.1 An Euclidean finite-difference scheme.

We consider problem (2.3)-(2.4) for the price an European Put option with strike K. We solve (2.3) in the bounded cylinder

$$]Ke^{-\mu}, Ke^{\mu}[\times] - \nu, \nu[\times]0, T[, \tag{3.1}$$

for some $\mu, \nu > 0$, subject to the following set of initial-boundary conditions: if f denotes the solution, we obviously impose the initial condition

$$f(S, D, 0) = (K - S)^+, \qquad \text{for } (S, D) \in]Ke^{-\mu}, Ke^{\mu}[\times] - \nu, \nu[. \tag{3.2}$$

Moreover we assume

$$f(Ke^{-\mu}, D, t) = K \quad \text{and} \quad f(Ke^{\mu}, D, t) = 0, \qquad \text{for } (D, t) \in] - \nu, \nu[\times]0, T[. \tag{3.3}$$

Finally, let $p_{BS}(S, \sigma, T - t, K)$ denote the Black-Scholes price of a Put option with time to expiry $T - t$, strike K, underlying price S and volatility σ; we assume that

$$f(S, \pm\nu, t) = p_{BS}(S, \sigma(\pm\nu), T - t, K) \qquad \text{for } (S, t) \in]Ke^{-\mu}, Ke^{\mu}[\times]0, T[. \tag{3.4}$$

We consider the following transformation which slightly simplifies the problem:

$$f(S, D, t) = Ku\left(\log\left(S/K\right), D, \lambda t\right). \tag{3.5}$$

Then equation (2.3) (for $r = 0$) becomes

$$Lu := \frac{\sigma^2(y)}{2\lambda}\left(\partial_{\theta\theta}u - \partial_\theta u\right) - y\partial_y u - \partial_t u = 0, \tag{3.6}$$

for $u = u(x, y, t)$ defined in the cylinder

$$Q =] - \mu, \mu[\times] - \nu, \nu[\times]0, \lambda T[. \tag{3.7}$$

In (3.6) $\partial_\theta u = \partial_x u + \partial_y u$ denotes the directional derivative of u with respect to $\theta = (1, 1, 0)$.

The initial-boundary conditions for u read as follows: the initial condition (3.2) corresponds to

$$u(x, y, 0) = (1 - e^x)^+, \qquad (x, y) \in [-\mu, \mu] \times [-\nu, \nu]. \tag{3.8}$$

Condition (3.3) becomes

$$u(-\mu, y, t) = 1, \qquad u(\mu, y, t) = 0, \qquad (y, t) \in] - \nu, \nu[\times]0, \lambda T[. \tag{3.9}$$

Moreover, in view of the classical Black-Scholes formula, (3.4) corresponds to

$$u(x, \pm\nu, t) = \Phi\left(\frac{\sigma^2(\pm\nu)t/(2\lambda) - x}{\sigma(\pm\nu)\sqrt{t/\lambda}}\right) - e^x\Phi\left(\frac{-\sigma^2(\pm\nu)t/(2\lambda) - x}{\sigma(\pm\nu)\sqrt{t/\lambda}}\right), \tag{3.10}$$

for $(x,t) \in]-\mu, \mu[\times]0, \lambda T[$, where $\Phi(\cdot)$ is the standard normal cumulative distribution function. We explicitly remark that the function u is independent of K, therefore by (3.5) it provides option prices for different strikes.

We consider an explicit finite-differences scheme on the uniform grid

$$G = \{(i\triangle_x, k\triangle_y, n\triangle_t) \mid i,k,n \in \mathbb{Z}\},$$

and we approximate L in (3.6) by the following discrete operator

$$L_G u(z) = \frac{\sigma^2(y)}{2\lambda}\left(D^2_{\theta,\triangle_x} u(z) - D_{\theta,\triangle_x} u(z)\right) - y D_{y,\triangle_x} u(z) - D_{t,\triangle_t} u(z), \tag{3.11}$$

where $z = (x,y,t)$ and

$$\begin{aligned}
D^2_{\theta,\triangle_x} u(z) &= \frac{u(x+\triangle_x, y+\triangle_x, t) - 2u(z) + u(x-\triangle_x, y-\triangle_x, t)}{\triangle_x^2},\\
D_{\theta,\triangle_x} u(z) &= \frac{u(x+\triangle_x, y+\triangle_x, t) - u(x-\triangle_x, y-\triangle_x, t)}{2\triangle_x},\\
D_{y,\triangle_x} u(z) &= \frac{u(z) - u(x, y - \operatorname{sign}(y)\triangle_x, t)}{\operatorname{sign}(y)\triangle_x},\\
D_{t,\triangle_t} u(z) &= \frac{u(x,y,t+\triangle_t) - u(z)}{\triangle_t}.
\end{aligned}$$

Operator L_G is well-defined on the grid G with $\triangle_x = \triangle_y$ and approximates L in the sense that

$$\|Lu - L_G u\|_{L^\infty(Q\cap G)} \le C(\triangle_x + \triangle_t),$$

for some positive constant C depending on the L^∞-norms of $\sigma, \partial^3_\theta u, \partial^4_\theta u, \partial^2_y u$ and $\partial^2_t u$ on Q. Note that, in view of Remark 2.1, ∂^4_θ has to be considered as a derivative of order twelve (in the intrinsic sense) whose existence is guaranteed assuming that σ has derivatives of order ten $\mathcal{K}$-Hölder continuous.

By standard arguments, we can prove the following convergence result.

Theorem 3.1 *Let u be the solution to the Cauchy-Dirichlet problem for* (3.6) *in the cylinder Q subject to conditions* (3.8)-(3.9)-(3.10)*. Let u_G denote the solution to the correspondent discrete problem for L_G in $Q \cap G$ with the same initial-boundary conditions. Assume $\Delta_x \le 2$ and the following stability condition:*

$$\triangle_t \le \frac{\lambda}{\lambda\eta\Delta_x + \sup\sigma^2}\triangle_x^2. \tag{3.12}$$

Then

$$\|u - u_G\|_{L^\infty(Q\cap G)} = O(\triangle_x), \qquad \text{as } \triangle_x \longrightarrow 0^+.$$

Remark 3.2 *An implicit method can be developed by using the backward differences*

$$\frac{u(z) - u(x,y,t-\triangle_t)}{\triangle_t} \tag{3.13}$$

instead of the forward differences $D_{t,\triangle_t}u(z)$ in L_G. Even if this approach has the advantage of being unconditionally stable (i.e. the time-step $\triangle_t$ can be chosen independently of $\triangle_x$), however it is very computationally expensive. Indeed the numerical computation of u_G requires, at each time-step, the solution of a linear system of order $(2I+1)^2$, where $I = 2\mu/\triangle_x$. This becomes onerous in term of memory and computational time when the spatial grid spacing diminishes.

3.2 An explicit Kolmogorov finite-difference scheme.

We approximate problem (2.5)-(2.8) by a Cauchy-Dirichlet problem for

$$\mathcal{L}u := a(\partial_{xx}u - \partial_x u) + x\partial_y u - \partial_\tau u = 0 \tag{3.14}$$

with a defined in (2.7), in the cylinder

$$Q = \{(x, y, \tau) \mid |x| < \mu,\ |y| < \nu \text{ and } 0 < \tau < \tau_0\}, \tag{3.15}$$

where $\mu, \nu > 0$ and $\tau_0 = 1 - e^{-\lambda T}$. By transformation (2.6), this corresponds to the initial-boundary value problem for (2.3) in the twisted region

$$\{(S, D, t) \mid Ke^{-\mu} < S < Ke^{\mu},\ 0 < t < T \text{ and } -\nu e^{\lambda t} < D - \log(S/K) < \nu e^{\lambda t}\}.$$

In the sequel we denote by

$$\partial_P Q = \partial Q \cap \{(x, y, \tau) \mid \tau < \tau_0\}$$

the parabolic boundary of Q.

We consider an explicit Kolmogorov scheme on the uniform grid

$$G = \{(i\triangle_x, k\triangle_y, n\triangle_\tau) \mid i, k, n \in \mathbb{Z}\}, \tag{3.16}$$

and we discretize $\mathcal{L}$ in (3.14) by approximating the main directional derivatives as follows

$$\partial_{xx}u(z) \quad \sim \quad D^2_{\triangle_x}u(z) = \frac{u(x+\triangle_x, y, t) - 2u(z) + u(x-\triangle_x, y, \tau)}{\triangle_x^2}, \tag{3.17a}$$

$$\partial_x u(z) \quad \sim \quad D_{\triangle_x}u(z) = \frac{u(x+\triangle_x, y, \tau) - u(x-\triangle_x, y, \tau)}{2\triangle_x}, \tag{3.17b}$$

$$Yu(z) \quad \sim \quad Y_{\triangle_\tau}u(z) = \frac{u(z) - u(x, y - x\triangle_\tau, \tau + \triangle_\tau)}{\triangle_\tau}. \tag{3.17c}$$

Position (3.17c) forces to set

$$\triangle_y = \triangle_x \triangle_\tau, \tag{3.18}$$

since in this case the discrete operator defined by

$$\mathcal{L}_G u = a(D^2_{\triangle_x}u - D_{\triangle_x}u) + Y_{\triangle_\tau}u = 0 \tag{3.19}$$

is well-defined on the grid G. Moreover $\mathcal{L}_G$ approximates $\mathcal{L}$ in the sense that

$$\|\mathcal{L}u - \mathcal{L}_G u\|_{L^\infty(Q\cap G)} \le C(\triangle_x^2 + \triangle_\tau), \tag{3.20}$$

for some positive constant C depending on the L^∞-norms of $a, \partial_x^3 u, \partial_x^4 u$ and $Y^2 u$ on Q. Therefore estimate (3.20) involves the intrinsic derivatives of u up to the fourth order: this should be compared with the analogous result for the Euclidean scheme in the previous subsection.

We first show a general result about the convergence of the scheme (3.17) and then address the problem of the specification of the initial-boundary conditions for (3.14). By Remark 3.5 below, it is possible to assume that the vertices $(\pm\mu, \pm\nu, 0)$ belong to the grid G so that the discrete Cauchy-Dirichlet problem is well-posed. Following [15], we prove the following

Lemma 3.3 (Maximum principle) *Assume $\Delta_x \le 2$ and the stability condition*

$$\triangle_\tau \le \frac{\triangle_x^2}{2 \sup a}. \tag{3.21}$$

Let v be defined in $G \cap \overline{Q}$ be such that

$$\mathcal{L}_G v \ge 0, \qquad \text{in } G \cap Q, \tag{3.22}$$

$$v \le 0, \qquad \text{in } G \cap \partial_P Q. \tag{3.23}$$

Then $v \le 0$ in $G \cap Q$.

Proof. Denoting

$$v_{i,k}^n = v(i\triangle_x, k\triangle_y, n\triangle_\tau),$$

we have that (3.22) is equivalent to

$$v_{i,k}^n \le v_{i,k+i}^{n-1}\left(1 - 2a_{i,k+i}^{n-1}\frac{\triangle_\tau}{\triangle_x^2}\right) + a_{i,k+i}^{n-1}\frac{\triangle_\tau}{\triangle_x^2}\left((1-\triangle_x/2)v_{i+1,k+i}^{n-1} + (1+\triangle_x/2)v_{i-1,k+i}^{n-1}\right). \tag{3.24}$$

The thesis follows by an elementary inductive argument. □

By means of the maximum principle it is standard to prove the following convergence result.

Theorem 3.4 *Let u (resp. u_G) denote the solution to the (discrete) Cauchy-Dirichlet problem for* (3.14) *(resp.* (3.19)*) in Q (resp. $Q \cap G$) with given initial-boundary conditions. Assume the stability condition* (3.21)*. Then*

$$\|u - u_G\|_{L^\infty(Q\cap G)} = O(\triangle_x^2), \qquad \text{as } \triangle_x \longrightarrow 0^+.$$

Remark 3.5 *It is easy to choose μ, ν, τ_0 and a grid G such that conditions* (3.18) *and* (3.21) *hold and the vertices $(\pm\mu, \pm\nu, 0)$ belong to G. Suppose that μ and τ_0 are given: we fix a natural number m and set $\nu = m\mu\tau_0$. Moreover we put $\triangle_x = \mu/I$ and $\triangle_\tau = \tau_0/N$ where $I, N \in \mathbb{N}$ and $N \ge \frac{2\tau_0 \sup a}{\triangle_x^2}$ so that the stability condition holds. Finally, we set $\triangle_y = \triangle_x\triangle_\tau$. Then we have*

$$G \cap \overline{Q} = \{(i\triangle_x, k\triangle_y, n\triangle_\tau) \mid |i| \le I,\ |k| \le mNI \ \text{ and } \ 0 \le n \le N\}. \tag{3.25}$$

We now consider the problem of the specification of the initial-boundary conditions. Clearly we impose the initial condition

$$u(x,y,0) = (1-e^x)^+, \qquad \text{for } \ x \in [-\mu,\mu], \ y \in [-\nu,\nu]. \tag{3.26}$$

Moreover we set

$$u(-\mu,y,\tau) = 1 \ \text{ and } \ u(\mu,y,\tau) = 0, \qquad \text{for } \ y \in]-\nu,\nu[, \ \tau \in]0,\tau_0[. \tag{3.27}$$

A reason for conditions (3.27) is given by the following proposition whose proof is postponed to the end of the subsection.

Proposition 3.6 *Let u be the bounded solution to the Cauchy problem* (2.5)-(2.8)*. Then*

$$\lim_{x\to-\infty} u(x,y,\tau) = 1 \qquad \textit{and} \qquad \lim_{x\to+\infty} u(x,y,\tau) = 0,$$

uniformly in $(y,\tau) \in \mathbb{R}\times[0,\tau_0]$.

Finally we show a simple way to avoid giving conditions on the lateral boundary $\{y=\pm\nu\}$ of Q. Let us consider the grid in (3.25). We first remark that for a solution v to $\mathcal{L}_G v = 0$ in $G\cap\overline{Q}$, the value $v^n_{i,k}$ only depends on $v^{n-1}_{i,k+i}$ and $v^{n-1}_{i\pm1,k+i}$ (see (3.24)). More generally, it is straightforward to determine the domain of dependence of the set of values

$$V_m = \{v^N_{i,k} \ | \ |i|\le I, \ |k| \le (m-1)NI\}:$$

indeed, if we put

$$k^{(n)} = \max\{|k| \ | \ V_m \text{ depends on } v^n_{i,k}\},$$

then, by (3.24), we have

$$k^{(n)} = k^{(n+1)} + I = \cdots = k^{(N)} + (N-n)I = I(mN-n).$$

In particular $k^{(0)} = mNI$, therefore V_m is independent on the value of v at the lateral boundary $y=\pm\nu$. Let us note that this is true for every refinement of the grid, that is for every choice of N and I. In view of these remarks, in order to approximate the solution $u(x,y,\tau_0)$ for $|y|\le(m-1)\mu\tau_0$, conditions on the lateral boundary $\{y=\pm\nu\}$ are superfluous. Alternatively, one can solve (3.14) in the prism

$$\{(x,y,\tau) \ | \ |x|<\mu, \ 0<\tau<\tau_0 \ \text{ and } \ |y|<\mu(m\tau_0-\tau)\}, \tag{3.28}$$

rather than in the whole cylinder Q.

Remark 3.7 *Although attractive from an analytical viewpoint, the explicit Kolmogorov scheme has a high computational cost. Indeed the stability condition* (3.21) *combined with assumption* (3.18) *implies $\triangle_y = O(\triangle_x^3)$ as $\triangle_x \longrightarrow 0$. Thus, for instance, in a grid with 10^2 $\triangle_x-$nodes we have about 10^4 $\triangle_\tau-$nodes and 10^6 $\triangle_y-$nodes. For this reason, in the next subsection, we propose a first order implicit scheme which combines the fine analytical properties of Kolmogorov schemes with the numerical efficiency.*

Let us recall that an explicit Kolmogorov difference scheme for the Hobson-Rogers model was first proposed in [4]. The starting point was the following PDE given in Sect.4.3 in [6]:

$$\frac{\sigma^2(x-y)}{2}(\partial_{xx}u - \partial_x u) + (x-y)\partial_y u - \partial_t u = 0. \tag{3.29}$$

Following the same approach used in [4], one may approximate the first order term $\partial_\theta u$, where $\theta = (0, x-y, -1)$, by the difference

$$\frac{u(z) - u(z + \Delta_t \theta)}{\Delta_t}.$$

This approach has the drawback that the two points z and $z + \Delta_t\theta$ cannot belong both to a uniform grid G like in (3.16): this is essentially due to the lack of a homogeneous structure for (3.29) analogous to that of the Kolmogorov equation (1.2). To overcome this problem, Di Francesco and Pascucci propose to approximate $\partial_\theta u$ by the "corrected difference"

$$\widetilde{Y}_G u(x,y,t) = \frac{u(x,y,t) - u(x, y + \Delta_t(x - \Delta_x[y/\Delta_x]), t - \Delta_t)}{\Delta_t}. \tag{3.30}$$

It is shown in Lemma 3.1 of [4] that $\widetilde{Y}_G$ is well defined on G when $\Delta_y = \Delta_x\Delta_t$. On the other hand, the "correction" introduces an additional error of order Δ_x. Specifically, they show that

$$|\partial_\theta u(x,y,t) - \widetilde{Y}_G u(x,y,t)| \le \Delta_x \|u_y\|_\infty + \Delta_t \|Y^2 u\|_\infty.$$

Here, since transformation (2.6) leads to a homogeneous Kolmogorov equation, we considerably improve the results in [4].

We close this subsection by proving Proposition 3.6.

Proof of Proposition 3.6. We only study the first limit, the second one being analogous. We note that for every $\varepsilon > 0$ there exists a positive constant $R = R(\varepsilon, \tau_0)$ such that

$$\int_{-\delta}^{+\infty}\int_{\mathbb{R}} \Gamma(x,y,\tau;\xi,\eta,0)\,d\eta\, d\xi \le \varepsilon, \tag{3.31}$$

for every $\delta > 0$, $x \le -\delta - R$ and $(y,\tau) \in \mathbb{R}\times[0,\tau_0]$. Indeed, by estimate (2.13), we have

$$\int_{-\delta}^{+\infty}\int_{\mathbb{R}} \Gamma(x,y,\tau;\xi,\eta,0)\,d\eta\, d\xi \le C \int_{-\delta}^{+\infty}\int_{\mathbb{R}} \Gamma^c_{\mathcal{K}}(x,y,\tau;\xi,\eta,0)\,d\eta\, d\xi$$

(integrating in the variable η over $\mathbb{R}$ and denoting by Γ^c the fundamental solution of the heat equation $c\partial_{xx} - \partial_\tau$)

$$= C\int_{-\delta}^{+\infty} \Gamma^c(x,\tau;\xi,0)\,d\eta \le \varepsilon$$

if $x \le -\delta - R$ and R is suitably large. This proves (3.31).

The thesis is a simple consequence of (3.31) and the fact that, by Theorem 2.2,

$$\iint_{\mathbb{R}^2} \Gamma(x,y,\tau;\xi,\eta,0)d\xi d\eta = 1, \qquad \forall x,y \in \mathbb{R},\ \tau > 0. \tag{3.32}$$

Indeed, by the representation formula (2.12) and by (3.32), we have

$$1 \geq u(z) = \iint_{\mathbb{R}^2} \Gamma(z;\xi,\eta,0)(1-\xi)^+ d\eta d\xi \geq$$

(if δ is suitably large)

$$\geq (1-\varepsilon) \int_{-\infty}^{-\delta} \int_{\mathbb{R}} \Gamma(z;\xi,\eta,0) d\eta d\xi \geq (1-\varepsilon)^2,$$

by (3.32) and (3.31) if $x \leq -\delta - R$. $\square$

3.3 An implicit Kolmogorov finite-difference scheme.

In view of the previous subsection, it is quite simple to derive an implicit Kolmogorov finite-difference scheme for equation (3.14). We set Q as in (3.15) and consider a grid as in Remark 3.5. Then using the same notations of Subsection 3.2, the discrete operator

$$\hat{\mathcal{L}}_G v_{i,k}^n = a_{i,k}^n \left(\frac{v_{i+1,k}^n - 2v_{i,k}^n + v_{i-1,k}^n}{\triangle_x^2} - \frac{v_{i+1,k}^n - v_{i-1,k}^n}{2\triangle_x} \right) + \frac{v_{i,k+i}^{n-1} - v_{i,k}^n}{\triangle_\tau} \tag{3.33}$$

is well-defined on G (remember that $\triangle_y = \triangle_x \triangle_\tau$) and approximates $\mathcal{L}$ in that

$$\|\mathcal{L}u - \mathcal{L}_G u\|_{L^\infty(Q\cap G)} \leq C(\triangle_x + \triangle_\tau).$$

We may consider the same set of initial-boundary conditions used in the previous subsection: in particular it is straightforward to show that conditions at the lateral boundary $\{y = \pm\nu\}$ *do not affect* the value of the solution in the prism (3.28).

Finally, by means of the following maximum principle it is not difficult to prove a convergence result for $\hat{\mathcal{L}}$ analogous to Theorem 3.4.

Lemma 3.8 (Maximum principle) *Suppose that*

$$\hat{\mathcal{L}}_G v \geq 0, \qquad \text{in } G \cap Q, \tag{3.34}$$

$$v \leq 0, \qquad \text{in } G \cap \partial_P Q. \tag{3.35}$$

Then $v \leq 0$ in $G \cap Q$.

Proof. By contradiction, suppose that $M = \max_{G\cap Q} v = v^n_{i,k} > 0$. Then denoting

$$A^n_{i,k} = 1 + 2a^n_{i,k}\frac{\triangle_\tau}{\triangle_x^2},$$

by (3.34), we have

$$MA^n_{i,k} = v^n_{i,k}A^n_{i,k} \leq v^{n-1}_{i,k+i} + a^n_{i,k}\frac{\triangle_\tau}{\triangle_x^2}\left((1-\triangle_x/2)v^n_{i+1,k} + (1+\triangle_x/2)v^n_{i-1,k}\right) \leq$$

(since $v^n_{i,k}$ is a maximum of v)

$$\leq MA^n_{i,k},$$

and we deduce that $v^{n-1}_{i,k+i} = v^n_{i+1,k} = v^n_{i-1,k} = M$. If necessary, we repeat this argument until we reach a point belonging to the parabolic boundary: then we clearly have a contradiction and the lemma is proved. □

Remark 3.9 *Notice that* $(v^n_{i,k})_{|i|\leq I}$ *can be computed independently for different values of* k. *Indeed, by* (3.33), *only points on the same line* $y = k\triangle_x$ *are related by the operator* $\hat{\mathcal{L}}_G$ *at time* $t = n\triangle_\tau$. *Thus, the computation of* $v^n_{i,k}$, *for* $|i| \leq I$ *and* $|k| \leq J$, *reduces to the solution of* $2J+1$ *independent tridiagonal linear systems of order* $2I+1$ *and it allows for a highly efficient implementation (see also Remark 3.2).*

4 Comparison of numerical approximations

We consider and compare the numerical performances of the finite-difference schemes introduced in the previous section. Specifically we consider the explicit Euclidean and implicit Kolmogorov schemes introduced in Sections 3.1 and 3.3, respectively.

In all the experiments, we have chosen

$$\sigma(D) = \eta\sqrt{1+\varepsilon D^2} \wedge M$$

as proposed in [6], where the parameters have been fixed to $\varepsilon = 5$, and $\lambda = 1$. Furthermore, we have chosen a unit strike $K = 1$, a zero interest rate $r = 0$ and a domain as in (3.1) with $\mu = 2.0$. For the Euclidean method, the time step length is chosen to ensure stability, while for the Kolmogorov scheme we have found experimentally that a number of steps $N = 50$ is good compromise between precision and execution speed.

Tables 1-4 report numerical results for the prices of Standard and Digital European Put options computed by the two methods when $S_0 = K = 1.0$ (at the money) and $D_0 = 0.1$. The values computed by the Euclidean and the Kolmogorov methods are shown in the second and fifth columns, respectively. The execution times in seconds of the two methods are reported in the fourth and last columns and refers to experiments performed on an Intel Pentium IV single CPU of 1.80Ghz of clock.

Values computed by a Monte-Carlo method are reported in the captions of the tables. In order to produce figures stable up to the fourth significant digit, $3 \cdot 10^7$ replications have been performed and this took about 34 minutes of cpu time. Relative percentage errors of the results produced by the two finite difference methods are shown in columns 3 and 6.

I	Euclidean Price	Err %	Exec. time	Kolmogorov Price	Err %	Exec. time
50	0.0406	0.00	0.2	0.0399	1.72	4.9
100	0.0409	0.73	1.6	0.0406	0.00	19.5
150	0.0409	0.73	7.0	0.0406	0.00	43.8
200	0.0409	0.73	19.6	0.0407	0.24	81.8
250	0.0409	0.73	93.7	0.0407	0.24	134.3

Table 1: Price of a European Put Option, with $S_0 = K = 1.0$, $D_0 = 0.1$, $T = 0.25$ and $\eta = 0.2$. Reference value: 0.0406

I	Euclidean Price	Err %	Exec. time	Kolmogorov Price	Err %	Exec. time
50	0.2699	1.62	5.1	0.2657	0.03	4.9
100	0.2690	1.28	52.2	0.2658	0.07	19.5
150	0.2687	1.17	230.8	0.2659	0.11	43.8
200	0.2686	1.13	706.7	0.2659	0.11	81.8
250	0.2685	1.09	1635.5	0.2659	0.11	134.3

Table 2: Price of a European Put Option, with $S_0 = K = 1.0$, $D_0 = 0.1$, $T = 0.75$ and $\eta = 0.7$. Reference value: 0.2656

Tables 1 and 3 report prices of options with small volatility and short time to maturity, that is $\eta = 0.2$ and $T = 0.25$. We also tested the robustness of the methods by considering the more computationally cumbersome problem of high volatility ($\mu = 0.7$) and a bigger time to maturity ($T = 0.75$): the results are showed in Tables 2 and 4.

I	Euclidean Price	Err %	Exec. time	Kolmogorov price	Err %	Exec. time
50	0.5520	4.56	0.2	0.4460	15.51	4.9
100	0.5116	3.08	1.6	0.4879	7.57	19.5
150	0.4983	5.60	7.0	0.5013	5.03	43.8
200	0.4917	6.85	19.6	0.5081	3.75	81.8
250	0.4878	7.59	93.7	0.5121	2.99	134.3

Table 3: Price of a European Digital Put Option, with $S_0 = K = 1.0$, $D_0 = 0.1$, $T = 0.25$ and $\eta = 0.2$. Reference value: 0.5279

The data presented in the tables confirm the theoretical results presented in the previous sections. In particular, notice that in the experiments with discontinuous payoff reported in Tables 3 and 4, the Euclidean method do not converge to the correct price due to the influence of the boundary data. As opposite, the Kolmogorov scheme demonstrates its robustness converging, even if not rapidly, to the solution.

Regards to the computational cost of the methods, it should be noticed that the number of steps (and thus the computational cost) required by the Euclidean scheme depends on the volatility as well as on the time to expiration. Then, for simple problems like the one

	Euclidean			Kolmogorov		
I	Price	Err %	Exec. time	Price	Err %	Exec. time
50	0.4678	28.60	5.1	0.6409	2.18	5.1
100	0.4640	29.18	60.6	0.6485	1.02	20.8
150	0.4628	29.36	243.5	0.6510	0.64	47.2
200	0.4622	29.45	706.7	0.6522	0.45	84.9
250	0.4619	29.50	1635.5	0.6530	0.33	148.5

Table 4: Price of a European Digital Put Option, with $S_0 = K = 1.0$, $D_0 = 0.1$, $T = 0.75$ and $\eta = 0.7$. Reference value: 0.6552

reported in Table 1, an Euclidean scheme is quite acceptable (even if it should be noted that, by reducing the grid spacing, we do not obtain any significant improvement of precision). On the contrary, the other examples show that the Euclidean scheme is more computationally expensive than the Implicit Kolmogorov and not reliable (let us also recall Remark 3.2 and that Implicit schemes of Euclidean type are particularly onerous). This is particularly apparent in Table 4 which shows a persistent error of nearly 30% in the Euclidean scheme, in addition to a high computational cost. This example enlightens the advantage of the Kolmogorov scheme in not requiring the boundary conditions on the lateral boundary $\{y = \pm\nu\}$. Experiments on a Crank-Nicholson method based on the Kolmgorov scheme have also been performed, but are not here reported because of its numerical unreliability when dealing with discontinuous payoff.

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