Gauge invariance and asymptotic behavior for the Ginzburg-Landau equations of superconductivity

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Abstract

In this paper we study the gauge-invariance of the time-dependent Ginzburg Landau equations through the introduction of a model which uses observable variables. Since different choices of gauge lead to a different representation of such variables, the classical formulations of the Ginzburg Landau model do not allow to establish the property of gauge-invariance. With a suitable decomposition of the unknown fields, we write the problem in terms of real variables and deduce some energy estimates which prove the existence of a maximal attractor for the system.

Keywords: Superconductivity, gauge-invariance, global attractor.
AMS subject classifications: 82D55, 35B41.

1 Introduction

This paper has two different aims. In the first part we examine the gauge invariance of the time-dependent Ginzburg-Landau equations, (also called Gor'kov-Eliashberg equations [6], [11]), which describe the behavior of a superconductor during the phase transition between the normal and the superconducting state. As already pointed out by several authors ([2], [7]), such equations are invariant up to a gauge transformation and the invariance of the model means that the physical problem cannot be affected by the particular choice of the gauge. However, the results in literature do not allow to establish this property. Indeed, even if it is possible to write the problem by means of observable variables (i.e. in a gauge-invariant form), an existence and uniqueness theorem of the solution of this system is not proved yet. Accordingly, it is not clear if another choice of gauge yields solutions which are different from a physical point of view. More precisely, in Section 2 we introduce a decomposition of the velocity of superconducting electrons and observe that the choice of the gauge in the classical formulations is equivalent to the choice of a particular decomposition. The lack of a theorem of uniqueness for the problem written by means of observable variables implies that, by changing the gauge, the velocity of superconducting electrons could assume a decomposition which leads to a different phase space. Therefore, since we cannot state the gauge invariance of the model, in the second part of the paper, we study the asymptotic behaviour of the solutions in the London gauge, although the long-time behavior has been studied also in [10] with

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the Lorentz gauge. The existence and uniqueness of the solution with the choice of London gauge has been proved in [12]. In the same paper the authors prove also the existence of the global attractor for the Ginzburg-Landau system with a technique which does not make use of energy estimates. In Section 4, we deduce some energy estimates which allow to prove the existence of the global attractor. The estimates are established for a system of real equations which is obtained by means of the decomposition of the observable variables and which is equivalent to the classical Gor'kov-Eliashberg system.

2 Superconductivity and gauge invariance of the Ginzburg-Landau equations

The most outstanding property of a superconductor is the complete disappearance of the electrical resistivity at some low *critical temperature* T_c, which is characteristic of the material. However, there exists a second effect which is equally meaningful. This phenomenon, called Meissner effect, is the perfect diamagnetism. In other words, the magnetic field is expelled from the superconductor, independently of whether the field is applied in the superconductive state (zero-field-cooled) or already in the normal state (field-cooled).

In the London theory [8], [9] and in the paper [4] it is assumed that the supercurrent $\mathbf{J}_s$ inside the superconductor is related to the *magnetic field* $\mathbf{H}$ by the constitutive equation

$$\nabla \times \Lambda \mathbf{J}_s = -\mu \mathbf{H} \tag{2.1}$$

where $\Lambda(x)$ is a scalar coefficient characteristic of the material and μ is the *magnetic permeability.* The equation (2.1) is able to describe both the effects of superconductivity, namely the complete disappearance of the electrical resistivity and the Meissner effect.

An important step in the phenomenological description of superconductivity was the Ginzburg-Landau theory ([5]), which describes the phase transition between the normal and the superconducting state.

Landau argued that this transition induces a sudden change in the symmetry of the material and suggested that the symmetry can be measured by a complex-valued parameter ψ, called order parameter. The physical meaning of ψ is specified by saying that $f^2 = |\psi|^2$ is the number density, n_s, of superconducting electrons. Hence $\psi = 0$ means that the material is in the normal state, i.e. $T > T_c$, while $|\psi| = 1$ corresponds to the state of a perfect superconductor ($T = 0$).

There must exist a relation between ψ and the absolute temperature T and this occurs through the free energy e. If the magnetic field is zero, at constant pressure and around the critical temperature T_c the free energy e_0 is written as

$$e_0 = -a(T)|\psi|^2 + b(T)|\psi|^4$$

where higher-order terms in $|\psi|^4$ are neglected, so that the model is valid around the critical temperature T_c for small values of $|\psi|$.

Suppose that the superconductor occupies a bounded domain Ω, with regular boundary $\partial\Omega$ and denote by $\mathbf{n}$ the unit outward normal to $\partial\Omega$. If a magnetic field occurs, then the free energy of the material is given by

$$\int_\Omega e(\psi, T, \mathbf{H}) dx = \int_\Omega [e_0(\psi, T) + \mu \mathbf{H}^2 + \frac{1}{2m_*}| - i\hbar\nabla\psi - e_*\mathbf{A}\psi|^2] dx - \int_{\partial\Omega} \mathbf{A} \times \mathbf{H}_{ex} \cdot \mathbf{n}\, da \tag{2.2}$$

where m_* is the mass of the superelectron and e_* is its effective charge, $\mathbf{A}$ is the vector potential related to $\mathbf{H}$ and $\hbar$ is Planck's constant. The vector $\mathbf{H}_{ex}$ represents the external magnetic field on the boundary $\partial\Omega$ and we suppose $\nabla \times \mathbf{H}_{ex} = \mathbf{0}$.

The generalization of the Ginzburg-Landau theory to the evolution problem was analyzed by Schmid [11], Gor'kov and Eliashberg [6] in the context of the BCS theory of superconductivity. Now the total current density $\mathbf{J}$ is given by $\mathbf{J} = \mathbf{J}_s + \mathbf{J}_n$, where $\mathbf{J}_n$ obeys the Ohm's law

$$\mathbf{J}_n = \sigma \mathbf{E},$$

while the supercurrent $\mathbf{J}_s$ satisfies the London equation (2.1). In order to describe the physical state of the evolution system, Gor'kov and Eliashberg consider three variables, the *wave function* ψ, the *vector* and *scalar potential* $\mathbf{A}$ and ϕ, which are related to the electrical and magnetic fields $\mathbf{E}, \mathbf{H}$ by means of the equations

$$\mathbf{E} = -\frac{\partial \mathbf{A}}{\partial t} + \nabla\phi \quad , \qquad \mu\mathbf{H} = \nabla \times \mathbf{A} \tag{2.3}$$

The evolution model of superconductivity is governed by the differential system ([6], [11])

$$\gamma\left(\frac{\partial \psi}{\partial t} - i\frac{e_*}{\hbar}\phi\psi\right) = -\frac{1}{2m_*}(i\hbar\nabla + e_*\mathbf{A})^2\psi + \alpha\psi - \beta\left|\psi\right|^2\psi \tag{2.4}$$

$$\sigma\left(\frac{\partial \mathbf{A}}{\partial t} - \nabla\phi\right) = -\frac{1}{\mu}\nabla \times \nabla \times \mathbf{A} + \mathbf{J}_s \tag{2.5}$$

with

$$\mathbf{J}_s = -\frac{i\hbar e_*}{2m_*}(\psi^*\nabla\psi - \psi\nabla\psi^*) - \frac{e_*^2}{m_*}\left|\psi\right|^2\mathbf{A} \tag{2.6}$$

and γ a suitable coefficient representing a relaxation time. The associated boundary conditions are given by

$$\left(i\hbar\nabla + e_*\mathbf{A}\right)\psi \cdot \mathbf{n}|_{\partial\Omega} = 0 \quad , \quad (\nabla \times \mathbf{A}) \times \mathbf{n}|_{\partial\Omega} = \mu\mathbf{H}_{ex} \times \mathbf{n} \tag{2.7}$$

The system (2.4)-(2.6) must be invariant under a gauge transformation

$$(\psi, \mathbf{A}, \phi) \longleftrightarrow (\psi e^{i\frac{e_*}{\hbar}\chi}, \mathbf{A} + \nabla\chi, \phi - \dot{\chi}) \tag{2.8}$$

where the gauge χ can be any smooth scalar function of (x, t).

Various gauges have been considered ([2], [7], [12], [15]). In the London gauge, χ is chosen so that $\nabla \cdot \mathbf{A} = 0$, $\mathbf{A} \cdot \mathbf{n}|_{\partial\Omega} = 0$. In the Lorentz gauge we have $\phi = -\frac{1}{\mu\sigma}\nabla \cdot \mathbf{A}$ and the boundary condition $\mathbf{A} \cdot \mathbf{n}|_{\partial\Omega} = 0$. Finally, in the zero electrical potential gauge we have $\phi = 0$. It is not possible to have both $\phi = 0$, and the London gauge simultaneously.

The gauge invariance of the system (2.4)-(2.6) has been stated in many papers, where it is emphasized that the choice of the gauge is technical and does not affect the physical meaning of the solutions. We will show that this problem is still open. To this aim, we observe that the system (2.4)-(2.6) can be written by means of the observable variables $f, \mathbf{J}_s, \mathbf{H}, \mathbf{E}$, which are necessarily independent by the choice of the gauge. Indeed from (2.4) we deduce the equation ([3])

$$\gamma\frac{\partial f}{\partial t} = \frac{\hbar^2}{2m_*}\triangle f - \left(\frac{e_*^2}{2m_*}\right)^{-1}\mathbf{J}_s^2 f^{-1} + \alpha f - \beta f^3 \tag{2.9}$$

and in view of (2.6) we obtain London's equation

$$\nabla \times \Lambda(f)\mathbf{J}_s = -\mu \mathbf{H} \tag{2.10}$$

where $\Lambda(f) = \frac{2m_*}{e_*^2} f^{-2}$.

Equation (2.5) is essentially Ampere's law

$$\nabla \times \mathbf{H} = \mathbf{J}_s + \mathbf{J}_n + \varepsilon \frac{\partial \mathbf{E}}{\partial t}$$

when $\frac{\partial \mathbf{E}}{\partial t}$ is supposed negligible, namely when we consider the quasi-steady approximation.

Finally, by substituting the relation (2.10) in Maxwell equation

$$\nabla \times \mathbf{E} = -\mu \frac{\partial \mathbf{H}}{\partial t} \tag{2.11}$$

we have

$$\frac{\partial \Lambda(f)\mathbf{J}_s}{\partial t} = \mathbf{E} - \nabla \phi_s \tag{2.12}$$

where $\phi_s(x,t)$ is a smooth scalar function. The equation (2.12) corresponds to the Euler equation for a non-viscous electronic liquid (see [9], pag. 59) "where ϕ_s is the thermodynamic potential per electron; a function, in particular, of the concentrations of the superelectrons".

In order to obtain the complete equivalence with the problem (2.4)-(2.6), "the pressure" ϕ_s has to be related to the $\nabla \cdot \mathbf{E}$ by means of the identity ([1])

$$\phi_s = \frac{\hbar^2 \sigma}{2m_* \gamma} \Lambda(f) \nabla \cdot \mathbf{E} \tag{2.13}$$

Hence, in the quasi-steady approximation, equations (2.9)-(2.12) can be written also in the new form

$$\gamma \frac{\partial f}{\partial t} = \frac{\hbar^2}{2m_*} \triangle f - \frac{e_*^2}{2m_*} \mathbf{p}_s^2 f + \alpha f - \beta f^3 \tag{2.14}$$

$$\frac{1}{\mu} \nabla \times \nabla \times \mathbf{p}_s + \Lambda^{-1}(f)\mathbf{p}_s + \sigma \mathbf{E} = 0 \tag{2.15}$$

$$\mathbf{E} = \frac{\partial \mathbf{p}_s}{\partial t} + \nabla \phi_s$$

where $\mathbf{p}_s = \Lambda(f)\mathbf{J}_s$ denotes the velocity of superelectrons.

Moreover by means of (2.13) and (2.15), we get

$$\nabla \cdot (\Lambda^{-1}(f)\mathbf{p}_s) = -\sigma \nabla \cdot \mathbf{E} = -\frac{2m_* \gamma}{\hbar^2} \Lambda^{-1}(f) \phi_s \tag{2.16}$$

Concerning the boundary conditions, we assume

$$\mathbf{E} \cdot \mathbf{n}|_{\partial \Omega} = 0 \tag{2.17}$$

Together with the conditions (2.7), the previous relation yields

$$\nabla f \cdot \mathbf{n}|_{\partial\Omega} = 0\ , \quad (\nabla \times \mathbf{p}_s) \times \mathbf{n}|_{\partial\Omega} = -\mu \mathbf{H}_{ex} \times \mathbf{n}\ , \quad f\mathbf{p}_s \cdot \mathbf{n}|_{\partial\Omega} = 0,\ f\nabla\phi_s \cdot \mathbf{n}|_{\partial\Omega} = 0 \tag{2.18}$$

The equivalence between the systems (2.4)-(2.7) and (2.14)-(2.18) holds only if we consider regular solutions. Concerning the weak solutions, we observe that the two different representations could lead to set the problem in different functional spaces. In order to examine such equivalence we will show how it is possible to obtain the original system (2.4)-(2.6) starting from the real equations (2.14)-(2.16). The method we will follow will be able to exploit the physical meaning of the choice of gauge.

The main assumption for this procedure is the decomposition of the velocity $\mathbf{p}_s$ in the form[1]

$$\mathbf{p}_s = -\mathbf{A} + \nabla\theta \tag{2.19}$$

where θ is an arbitrary scalar field and $\mathbf{A}$ satisfies the relations (2.3).

The system (2.14)-(2.16) can be written in non dimensional form as

$$\dot{f} - \frac{1}{k^2}\triangle f + (f^2-1)f + f|\mathbf{p}_s|^2 = 0, \tag{2.20}$$

$$\eta(\dot{\mathbf{p}}_s + \nabla\phi_s) + \nabla\times\nabla\times\mathbf{p}_s + f^2\mathbf{p}_s = 0 \tag{2.21}$$

$$k^2 f\phi_s + f\nabla\cdot\mathbf{p}_s + 2\nabla f\cdot\mathbf{p}_s = 0 \tag{2.22}$$

and, by using (2.19), we get

$$\dot{f} - \frac{1}{k^2}\triangle f + (f^2-1)f + f|\mathbf{A}-\nabla\theta|^2 = 0, \tag{2.23}$$

$$\eta(\dot{\mathbf{A}} - \nabla\phi) + \nabla\times\nabla\times\mathbf{A} + f^2(\mathbf{A}-\nabla\theta) = 0 \tag{2.24}$$

$$k^2 f(\dot{\theta} - \phi) + f\nabla\cdot(\mathbf{A}-\nabla\theta) + 2\nabla f\cdot(\mathbf{A}-\nabla\theta) = 0 \tag{2.25}$$

where

$$\phi = \dot{\theta} + \phi_s\ , \tag{2.26}$$

By means of the decomposition (2.19) we obtain the original Gor'kov-Eliashberg system

$$\dot{\psi} - ik\phi\psi + \left(\frac{i}{k}\nabla + \mathbf{A}\right)^2\psi - (1-|\psi|^2)\psi = 0 \tag{2.27}$$

$$\eta(\dot{\mathbf{A}} - \nabla\phi) + \nabla\times\nabla\times\mathbf{A} = -\frac{i}{2}\left[\psi^*(\nabla\psi - i\mathbf{A}\psi) - \psi(\nabla\psi^* + i\mathbf{A}\psi^*)\right] \tag{2.28}$$

In fact, if we put

$$\psi = f\, e^{ik\theta}$$

then by (2.23), (2.25) we have (2.27) and by (2.24) we obtain (2.28). Therefore the systems (2.27)-(2.28), (2.20)-(2.22) and (2.23)-(2.25) are formally equivalent.

[1] In order to simplify our notations, hereafter we consider the Ginzburg-Landau equations in a non dimensional form. Moreover we denote by a superimposed dot the partial derivative with respect to the variable t.

From a physical point of view, the representation (2.19) means that $\mathbf{p}_s$ is decomposed as the sum of an irrotational field and a vector $\mathbf{A}$, whose definition depends on the choice of the gauge. For instance, if we consider London gauge, $\mathbf{A}$ will be a solenoidal field. Accordingly, the decomposition (2.19) is not unique. In order to obtain the uniqueness of the solution, we need to choose a decomposition for $\mathbf{p}_s$, which corresponds to fix the gauge in the classical Gor'kov-Eliashberg system. Hence the properties of the vector $\mathbf{p}_s$ could change when we choose a differnt gauge. As already pointed out in the Introduction, the invariance up to gauge transformations can be estabilished once we have proved an existence and uniqueness theorem for the system (2.20)-(2.22) with appropriate initial and boundary conditions. In this way the solution of the problem cannot be affected by the choice of the decomposition (2.19). Unfortunately, such a result seems not to have been proved in literature. Thus, the gauge invariance of the Ginzburg-Landau model remains an open problem.

For this reason, in the following we will perform a choice of the decomposition (2.19), namely we will suppose

$$\nabla\cdot\mathbf{A}=0, \qquad \mathbf{A}\cdot\mathbf{n}|_{\partial\Omega}=0 \qquad \int_\Omega \phi dx=0 \tag{2.29}$$

Accordingly, we restrict our attention to the system

$$\dot{f}-\frac{1}{k^2}\triangle f+(f^2-1)f+f|\mathbf{A}-\nabla\theta|^2=0, \tag{2.30}$$

$$\eta(\dot{\mathbf{A}}-\nabla\phi)+\nabla\times\nabla\times\mathbf{A}+f^2(\mathbf{A}-\nabla\theta)=0 \tag{2.31}$$

$$k^2 f(\dot{\theta}-\phi)+f\nabla\cdot(\mathbf{A}-\nabla\theta)+2\nabla f\cdot(\mathbf{A}-\nabla\theta)=0, \tag{2.32}$$

and associate the corresponding boundary conditions

$$\nabla f\cdot\mathbf{n}|_{\partial\Omega}=0\ ,\ (\nabla\times\mathbf{A})\times\mathbf{n}|_{\partial\Omega}=\mathbf{H}_{ex}\times\mathbf{n}\ ,\ f\nabla\theta\cdot\mathbf{n}|_{\partial\Omega}=0,\ \nabla\phi\cdot\mathbf{n}|_{\partial\Omega}=0 \tag{2.33}$$

Moreover, by taking the divergence of (2.31) and using $(2.29)_1$, we obtain the following equation

$$\eta\triangle\phi-\nabla\cdot[f^2(\mathbf{A}-\nabla\theta)]=0 \tag{2.34}$$

Hence, the equation (2.32) yields

$$\eta\triangle\phi+k^2f^2(\dot{\theta}-\phi)=0 \tag{2.35}$$

3 Existence, uniqueness and properties of solutions

With different choices of gauge, existence and uniqueness results have been proved for the system (2.27)-(2.28) with the initial and boundary conditions

$$\psi(x,0)=\psi_0(x), \qquad \mathbf{A}(x,0)=\mathbf{A}_0(x), \tag{3.1}$$

$$\nabla\psi\cdot\mathbf{n}|_{\partial\Omega}=0 \qquad (\nabla\times\mathbf{A})\times\mathbf{n}|_{\partial\Omega}=-\mathbf{H}_{ex}\times\mathbf{n}|_{\partial\Omega} \tag{3.2}$$

$$\mathbf{A}\cdot\mathbf{n}|_{\partial\Omega}=0, \qquad \nabla\phi\cdot\mathbf{n}|_{\partial\Omega}=0 \tag{3.3}$$

We recall here some results proved in [12] and [14] which make use of London gauge. In order to obtain a precise formulation of the problem we introduce the following functional space

$$\mathbf{V}_0=\{\mathbf{A}\in H^1(\Omega):\nabla\cdot\mathbf{A}=0,\ \mathbf{A}\cdot\mathbf{n}|_{\partial\Omega}=0\}$$

Moreover we denote by $\|\cdot\|_p$ and $\|\cdot\|_{H^s}$ the norms in $L^p(\Omega)$ and $H^s(\Omega)$ respectively. For each $\mathbf{A} \in \mathbf{V}_0$, the inequalities

$$\begin{aligned} \|\mathbf{A}\|_{H^1} &\leq K_1 \|\nabla \times \mathbf{A}\|_2 && (3.4) \\ \|\mathbf{A}\|_{H^{1/2}(\partial\Omega)} &\leq K_2 \|\nabla \times \mathbf{A}\|_2 && (3.5) \end{aligned}$$

hold with K_1, K_2 positive constants depending on the domain Ω.

The following theorem, proved in [12], ensures the well posedness of the problem.

Theorem 3.1 *If* $(\psi_0, \mathbf{A}_0) \in H^1(\Omega) \times \mathbf{V}_0$, *there exists a unique solution* $(\psi, \mathbf{A})$ *of the problem (2.27)-(2.28) with boundary and initial conditions (3.1)-(3.3) such that* $\psi \in L^2(0,T; H^2(\Omega)) \cap C(0,T; H^1(\Omega))$, $\mathbf{A} \in L^2(0,T; \mathbf{V}_0 \cap H^2(\Omega)) \cap C(0,T; \mathbf{V}_0)$.

In view of the equivalence between the systems (2.27)-(2.28) and (2.30)-(2.32), we can obtain an existence and uniqueness theorem for the problem (2.27)-(3.3), by writing the functional spaces of the Theorem 3.1 in terms of the variables $f, \nabla\theta, \mathbf{A}, \phi$.

We conclude this section by showing a property of the solutions of the Ginzburg-Landau equations, which will be useful for the proof of the estimates in the following section.

Proposition 3.1 *If* $(f, \mathbf{p}_s, \phi_s)$ *is a solution such that* $f_0(x)^2 \leq 1$ *almost everywhere in* Ω, *then* $f(x,t)^2 \leq 1$ *a.e. in* $\Omega \times [0,T]$.

By multiplying the equation (2.20) by f we obtain

$$\frac{\partial}{\partial t}\frac{1}{2} f^2 + \frac{1}{k^2}|\nabla f|^2 - \frac{1}{2k^2}\triangle f^2 + (f^2-1)^2 + (f^2-1) + f^2 \mathbf{p}_s^2 = 0,$$

so that

$$\frac{\partial}{\partial t}(f^2-1) - \frac{1}{k^2}\triangle(f^2-1) + 2(f^2-1) \leq 0,$$

Now let us multiply the previous inequality by $h = (f^2-1)_+ = \max\{f^2-1, 0\}$. In this way we deduce

$$\frac{\partial}{\partial t}\frac{1}{2}h^2 - \frac{1}{k^2}h\triangle h + 2h^2 \leq 0,$$

Hence, by integrating on Ω, we obtain

$$\frac{\partial}{\partial t}\frac{1}{2}\|h\|_2^2 + \frac{1}{k^2}\|\nabla h\|_2^2 + 2\|h\|_2^2 \leq 0,$$

The assumption $f_0(x)^2 \leq 1$, allows to conclude that

$$\frac{1}{2}\|h\|_2^2 + \int_0^t \left[\frac{1}{k^2}\|\nabla h\|_2^2 + 2\|h\|_2^2\right] d\tau \leq 0,$$

for each $t \in [0,T]$, so that $f^2 \leq 1$ almost everywhere in $\Omega \times [0,T]$. □

4 Energy estimates

In this section we examine the asymptotic behavior of the solution of the Ginzburg-Landau system. To this end, we will define an energy functional $\mathcal{E}_0$ and prove the inequality which guarantees the existence of an absorbing set for the system. Let

$$\mathcal{E}_0(f,\mathbf{p}_s) = \frac{1}{2}\int_\Omega \left[\frac{1}{k^2}|\nabla f|^2 + \frac{1}{2}(f^2-1)^2 + |\nabla\times\mathbf{p}_s|^2 + f^2|\mathbf{p}_s|^2\right] dx \tag{4.1}$$

the energy associated to the system (2.20)-(2.22). By means of the decomposition (2.19) we can express the energy functional (4.1) in terms of the variables $(f,\nabla\theta,\mathbf{A})$, namely

$$\mathcal{E}_0(f,\nabla\theta,\mathbf{A}) = \frac{1}{2}\int_\Omega \left[\frac{1}{k^2}|\nabla f|^2 + \frac{1}{2}(f^2-1)^2 + |\nabla\times\mathbf{A}|^2 + f^2|\mathbf{A}-\nabla\theta|^2\right] dx$$

Moreover, we observe that $\mathcal{E}_0$ can be written as a function of the variables $(\psi,\mathbf{A})$ in the form

$$\mathcal{E}_0(\psi,\mathbf{A}) = \frac{1}{2}\int_\Omega \left[\left|(\frac{i}{k}\nabla+\mathbf{A})\psi\right|^2 + \frac{1}{2}(|\psi|^2-1)^2 + |\nabla\times\mathbf{A}|^2\right] dx$$

Note that $\mathcal{E}_0(\psi,\mathbf{A})$ has to be invariant up to gauge transformations of the form (2.8), since the energy depends on the observable variables $(f,\mathbf{p}_s)$ through the relation (4.1).

Theorem 4.1 *If the initial data satisfy $\mathcal{E}_0(f_0,\nabla\theta_0,\mathbf{A}_0)\le M$, then there exists a constant Γ, depending on Ω and $\mathbf{H}_{ex}$, such that for each $\Gamma'>\Gamma$, $\mathcal{E}_0(f,\nabla\theta,\mathbf{A})\le\Gamma'$ holds for $t>t_0$, where t_0 depends on M and $\Gamma'-\Gamma$.*

Proof. Henceforth, we denote by c_j, $j\in\mathbb{N}$, an arbitrary positive constant. By multiplying the equation (2.30) by $\dot f + c_1 f$, integrating on Ω and keeping $(2.33)_1$ into account, we obtain the equation

$$\begin{aligned}&\int_\Omega \left[\dot f^2 + \frac{1}{k^2}\nabla f\cdot\nabla\dot f + f\dot f|\mathbf{A}-\nabla\theta|^2 + (f^3-f)\dot f\right] dx\\ &+c_1\int_\Omega\left[f\dot f + \frac{1}{k^2}|\nabla f|^2 + f^2|\mathbf{A}-\nabla\theta|^2 + (f^2-1)f^2\right] dx = 0\end{aligned}$$

Hence

$$\begin{aligned}&\frac{d}{dt}\int_\Omega\left[\frac{1}{2k^2}|\nabla f|^2 + \left(\frac{f^4}{4}-\frac{f^2}{2}\right) + \frac{c_1}{2}f^2\right] dx\\ &+\int_\Omega\left[\dot f^2 + f\dot f|\mathbf{A}-\nabla\theta|^2 + \frac{c_1}{k^2}|\nabla f|^2 + c_1(f^4-f^2) + c_1 f^2|\mathbf{A}-\nabla\theta|^2\right] dx = 0\end{aligned} \tag{4.2}$$

Similarly, by multiplying the equation (2.31) by $\dot{\mathbf{A}} + c_2\mathbf{A}$, integrating by parts and using $(2.33)_2$, we have

$$\begin{aligned}&\frac{d}{dt}\int_\Omega\left[\frac{1}{2}|\nabla\times\mathbf{A}|^2 + \frac{\eta c_2}{2}|\mathbf{A}|^2\right] dx + \frac{d}{dt}\int_{\partial\Omega}\mathbf{A}\cdot\mathbf{H}_{ex}\times\mathbf{n}da\\ &+\int_\Omega\left[\eta|\dot{\mathbf{A}}|^2 + f^2(\mathbf{A}-\nabla\theta)\cdot\dot{\mathbf{A}} + c_2|\nabla\times\mathbf{A}|^2 + c_2 f^2(\mathbf{A}-\nabla\theta)\cdot\mathbf{A}\right] dx + \int_{\partial\Omega}c_2\mathbf{A}\cdot\mathbf{H}_{ex}\times\mathbf{n}da = 0\end{aligned} \tag{4.3}$$

Note that, in the previous equation the term involving $\nabla\phi$ vanishes as a consequence of $(2.29)_1$.

Finally, if we multiply equations (2.32) and (2.35) by $f\dot{\theta}$ and $-\phi$ respectively and integrate on Ω, we obtain the relations

$$\int_\Omega \left[k^2 f^2 \dot{\theta}^2 - k^2 f^2 \phi\dot{\theta} - f^2(\mathbf{A}-\nabla\theta)\cdot\nabla\dot{\theta}\right] dx = 0 \tag{4.4}$$

$$\int_\Omega \left[\eta|\nabla\phi|^2 - k^2 f^2 \phi\dot{\theta} + k^2 f^2 \phi^2\right] dx = 0 \tag{4.5}$$

where the boundary integrals vanish in view of $(2.33)_3$ and $(2.33)_4$.

The equations (4.2)-(4.5) yield

$$\begin{aligned}
&\frac{d}{dt}\int_\Omega \left[\frac{1}{2k^2}|\nabla f|^2 + \frac{1}{4}(f^2-1)^2 + \frac{c_1}{2}f^2 + \frac{1}{2}|\nabla\times\mathbf{A}|^2 + \frac{\eta c_2}{2}|\mathbf{A}|^2 + \frac{1}{2}f^2|\mathbf{A}-\nabla\theta|^2\right] dx \\
&+\frac{d}{dt}\int_{\partial\Omega} \mathbf{A}\cdot\mathbf{H}_{ex}\times\mathbf{n}da + \int_\Omega \left[\frac{c_1}{k^2}|\nabla f|^2 + c_1(f^4-f^2) + c_2|\nabla\times\mathbf{A}|^2 + c_1 f^2|\mathbf{A}-\nabla\theta|^2\right] dx \\
&+c_2\int_{\partial\Omega} \mathbf{A}\cdot\mathbf{H}_{ex}\times\mathbf{n}da + \int_\Omega \left[\dot{f}^2 + k^2 f^2(\dot{\theta}-\phi)^2 + \eta\dot{\mathbf{A}}^2 + \eta|\nabla\phi|^2\right] dx \\
&= -\int_\Omega c_2 f^2(\mathbf{A}-\nabla\theta)\cdot\mathbf{A}dx
\end{aligned} \tag{4.6}$$

Let us introduce the functional

$$\begin{aligned}
\mathcal{F} &= \frac{1}{2}\int_\Omega \left[\frac{1}{k^2}|\nabla f|^2 + \frac{1}{2}(f^2-1)^2 + c_1 f^2 + |\nabla\times\mathbf{A}|^2 + \eta c_2|\mathbf{A}|^2 + f^2|\mathbf{A}-\nabla\theta|^2\right] dx \\
&+ \int_{\partial\Omega} \mathbf{A}\cdot\mathbf{H}_{ex}\times\mathbf{n}da + \frac{K_2^2}{2}\|\mathbf{H}_{ex}\times\mathbf{n}\|^2_{H^{-1/2}(\partial\Omega)}
\end{aligned}$$

where the constant K_2 is defined in (3.5). Note that $\mathcal{F}$ is positive definite since the relation (3.5) implies

$$\begin{aligned}
\int_{\partial\Omega} \mathbf{A}\cdot\mathbf{H}_{ex}\times\mathbf{n}da &\geq -\|\mathbf{A}\times\mathbf{n}\|_{H^{1/2}(\partial\Omega)}\|\mathbf{H}_{ex}\times\mathbf{n}\|_{H^{-1/2}(\partial\Omega)} \geq -K_2\|\nabla\times\mathbf{A}\|_2\,\|\mathbf{H}_{ex}\times\mathbf{n}\|_{H^{-1/2}(\partial\Omega)} \\
&\geq -\frac{1}{2}\|\nabla\times\mathbf{A}\|_2^2 - \frac{K_2^2}{2}\|\mathbf{H}_{ex}\times\mathbf{n}\|^2_{H^{-1/2}(\partial\Omega)}
\end{aligned}$$

Therefore $\mathcal{F} \geq \frac{1}{2}\mathcal{E}_0 \geq 0$.

On the other hand, the functional $\mathcal{F}$ can be written as

$$\mathcal{F} = \mathcal{E}_0 + \int_\Omega [c_1 f^2 + \eta c_2|\mathbf{A}|^2]dx + \int_{\partial\Omega} \mathbf{A}\cdot\mathbf{H}_{ex}\times\mathbf{n}da + \frac{K_2^2}{2}\|\mathbf{H}_{ex}\times\mathbf{n}\|^2_{H^{-1/2}(\partial\Omega)},$$

so that

$$\begin{aligned}
\mathcal{F} \leq \mathcal{E}_0 + \int_\Omega c_1(f^2-1)dx + \eta c_2 K_1\|\nabla\times\mathbf{A}\|_2^2 + K_2\|\nabla\times\mathbf{A}\|_2\,\|\mathbf{H}_{ex}\times\mathbf{n}\|_{H^{-1/2}(\partial\Omega)} \\
+\frac{K_2^2}{2}\|\mathbf{H}_{ex}\times\mathbf{n}\|^2_{H^{-1/2}(\partial\Omega)} + c_1\mathrm{vol}(\Omega)
\end{aligned}$$

Therefore, we can prove the existence of two positive constants C_1, C_2, depending on $\mathbf{H}_{ex}$ and Ω, such that

$$\frac{1}{2}\mathcal{E}_0 \leq \mathcal{F} \leq C_1\mathcal{E}_0 + C_2 \tag{4.7}$$

The relation (4.6) yields

$$\begin{aligned}
&\frac{d}{dt}\mathcal{F} + \int_\Omega \left[\frac{c_1}{k^2}|\nabla f|^2 + c_1(f^2-1)^2 + c_1 f^2 + c_2|\nabla\times\mathbf{A}|^2 + c_3|\mathbf{A}|^2 + c_1 f^2|\mathbf{A}-\nabla\theta|^2\right] dx \\
&+ \int_{\partial\Omega} c_2\mathbf{A}\cdot\mathbf{H}_{ex}\times\mathbf{n}da + \frac{K_2^2}{2}\|\mathbf{H}_{ex}\times\mathbf{n}\|^2_{H^{-1/2}(\partial\Omega)} + \int_\Omega \left[\dot{f}^2 + k^2 f^2(\dot{\theta}-\phi)^2 + \eta|\dot{\mathbf{A}}|^2 + \eta|\nabla\phi|^2\right] dx \\
&= \int_\Omega \left[c_3|\mathbf{A}|^2 - c_2 f^2(\mathbf{A}-\nabla\theta)\cdot\mathbf{A}\right] dx + \frac{K_2^2}{2}\|\mathbf{H}_{ex}\times\mathbf{n}\|^2_{H^{-1/2}(\partial\Omega)} + c_1 k^2 \mathrm{vol}(\Omega)
\end{aligned} \tag{4.8}$$

Concerning the right-hand side, observe that

$$\begin{aligned}
I_\Omega := \int_\Omega \left[c_3|\mathbf{A}|^2 - c_2 f^2(\mathbf{A}-\nabla\theta)\cdot\mathbf{A}\right] dx &\leq c_3\|\mathbf{A}\|_2^2 + c_2\|f(\mathbf{A}-\nabla\theta)\|_2\,\|f\mathbf{A}\|_2 \\
&\leq K_1 c_3\|\nabla\times\mathbf{A}\|_2^2 + c_2\left(\frac{1}{2c_4}\|f(\mathbf{A}-\nabla\theta)\|_2^2 + \frac{c_4}{2}\|f\mathbf{A}\|_2^2\right)
\end{aligned}$$

Moreover, in view of Proposition 3.1, we have

$$\|f\mathbf{A}\|_2^2 \leq \|\mathbf{A}\|_2^2 \leq K_1\|\nabla\times\mathbf{A}\|_2^2,$$

so that with the choices of $c_4 = \frac{1}{2K_1}$, $c_2 = c_1 c_4$, $c_3 = \frac{c_2}{4K_1}$, we obtain

$$I_\Omega \leq \frac{c_2}{2}\|\nabla\times\mathbf{A}\|_2^2 + \frac{c_1}{2}\|f(\mathbf{A}-\nabla\theta)\|_2^2.$$

Substitution in (4.8), leads to the inequality

$$\begin{aligned}
&\frac{d}{dt}\mathcal{F} + \int_\Omega \left[\frac{c_1}{k^2}|\nabla f|^2 + c_1(f^2-1)^2 + c_1 f^2 + \frac{c_2}{2}|\nabla\times\mathbf{A}|^2 + c_3|\mathbf{A}|^2 + \frac{c_1}{2} f^2|\mathbf{A}-\nabla\theta|^2\right] dx \\
&+ \int_{\partial\Omega} c_2\mathbf{A}\cdot\mathbf{H}_{ex}\times\mathbf{n}da + \frac{K_2^2}{2}\|\mathbf{H}_{ex}\times\mathbf{n}\|^2_{H^{-1/2}(\partial\Omega)} + \int_\Omega \left[\dot{f}^2 + k^2 f^2(\dot{\theta}-\phi)^2 + \eta|\dot{\mathbf{A}}|^2 + \eta|\nabla\phi|^2\right] dx \\
&\leq C
\end{aligned}$$

where

$$C = \frac{K_2^2}{2}\|\mathbf{H}_{ex}\times\mathbf{n}\|^2_{H^{-1/2}(\partial\Omega)} + c_1\mathrm{vol}(\Omega)$$

By putting $\lambda = 2\min\left\{c_1, \frac{c_2}{2}, \frac{c_3}{\eta c_2}, 1\right\}$, we have proved the inequality

$$\frac{d}{dt}\mathcal{F} + \lambda\mathcal{F} + \int_\Omega \left[\dot{f}^2 + k^2 f^2(\dot{\theta}-\phi)^2 + \eta|\dot{\mathbf{A}}|^2 + \eta|\nabla\phi|^2\right] dx \leq C \tag{4.9}$$

Hence

$$\frac{d}{dt}\mathcal{F}+\lambda\mathcal{F}\le C,$$

The application of Gronwall lemma yields

$$\mathcal{F}(t)\le\mathcal{F}(0)e^{-\lambda t}+\frac{C}{\lambda}\left(1-e^{-\lambda t}\right)\le\mathcal{F}(0)e^{-\lambda t}+\frac{C}{\lambda}\,.$$

Therefore, in view of the relation (4.7) we obtain the inequality

$$\mathcal{E}_0(t)\le 2\mathcal{F}(t)\le 2C_1\mathcal{E}_0(0)e^{-\lambda t}+\Gamma$$

where $\Gamma=2C_1+\frac{C}{\lambda}$. The assumption on the initial data allows to prove the inequality

$$\mathcal{E}_0(t)\le 2C_1Me^{-\lambda t}+\Gamma$$

Hence, for each $\Gamma'>\Gamma$, the inequality $\mathcal{E}_0(t)\le\Gamma'$ holds if $t>t_0=\max\left\{0,\frac{1}{\lambda}\log\frac{2C_1M}{\Gamma'-\Gamma}\right\}$. □

5 Higher-order energy estimates

We introduce now the higher-order energy functional defined as

$$\mathcal{E}_1(\psi,\mathbf{A})=\frac{1}{2}\int_\Omega\left[|(\frac{i}{k}\nabla+\mathbf{A})^2\psi|^2+|\nabla\times\nabla\times\mathbf{A}|^2\right]dx$$

Like the functional $\mathcal{E}_0$, the energy $\mathcal{E}_1$ can be written by means of the variables $(f,\nabla\theta,\mathbf{A})$ as

$$\mathcal{E}_1(f,\nabla\theta,\mathbf{A})=\frac{1}{2}\int_\Omega\left[(-\frac{1}{k^2}\triangle f+f|\mathbf{A}-\nabla\theta|^2)^2+(-\frac{1}{k}f\triangle\theta+\frac{2}{k}\nabla f\cdot(\mathbf{A}-\nabla\theta))^2+|\nabla\times\nabla\times\mathbf{A}|^2\right]dx \tag{5.1}$$

or by means of $(f,\mathbf{p}_s)$ as

$$\mathcal{E}_1(f,\mathbf{p}_s)=\frac{1}{2}\int_\Omega\left[(-\frac{1}{k^2}\triangle f+f\mathbf{p}_s^2)^2+(\frac{1}{k}f\nabla\cdot\mathbf{p}_s+\frac{2}{k}\nabla f\cdot\mathbf{p}_s)^2+|\nabla\times\nabla\times\mathbf{p}_s|^2\right]dx$$

We prove now some energy estimates for the functional (5.1). In order to simplify our notations we define

$$P=-\frac{1}{k^2}\triangle f+f|\mathbf{A}-\nabla\theta|^2,\qquad Q=-\frac{1}{k}f\triangle\theta+2\nabla f\cdot(\mathbf{A}-\nabla\theta)\,. \tag{5.2}$$

Moreover we denote by $c_j,\ j\in\mathbb{N}$, a generic positive constant.

By multiplying the equation (2.30) by $\dot P+c_1P-k\dot\theta Q$ and integrating in Ω, we obtain

$$\int_\Omega\left[\frac{d}{dt}\frac{P^2}{2}+c_1P^2+\dot f\dot P+f(f^2-1)[\dot P+c_1P-k\dot\theta Q]+c_1\dot fP-k\dot f\dot\theta Q-k\dot\theta PQ\right]dx=0 \tag{5.3}$$

Similarly, by multiplying (2.32) by $\dot{Q} + c_1 Q + k\dot{\theta} P$, we have

$$\int_\Omega \left[\frac{d}{dt}\frac{Q^2}{2} + c_1 Q^2 + f(\dot{\theta} - \phi)\dot{Q} + c_1 f(\dot{\theta} - \phi)Q + f(\dot{\theta} - \phi)\dot{\theta} P + \dot{\theta} PQ \right] dx = 0$$

Now we consider the equation (2.35) and multiply it by $\triangle\phi$, obtaining

$$\int_\Omega \left[\frac{\eta}{k^2}(\triangle\phi)^2 + f^2(\dot{\theta} - \phi)\triangle\phi \right] dx = 0 \tag{5.4}$$

The relations (5.3)-(5.4) yield

$$\frac{1}{2}\frac{d}{dt}\int_\Omega \left(P^2 + Q^2\right) dx + \int_\Omega \left[c_1 P^2 + c_1 Q^2 + \frac{\eta}{k^2}(\triangle\phi)^2 \right] dx + I_1 + I_2 = 0 \tag{5.5}$$

where I_1 and I_2 are defined as

$$\begin{aligned} I_1 &= \int_\Omega \left[\dot{f}\dot{P} + c_1 \dot{f} P - k\dot{f}\dot{\theta} Q + kf(\dot{\theta} - \phi)\dot{Q} + c_1 kf(\dot{\theta} - \phi)Q + k^2 f(\dot{\theta} - \phi)\dot{\theta} P + f^2(\dot{\theta} - \phi)\triangle\phi \right] dx \\ I_2 &= \int_\Omega f(f^2 - 1)[\dot{P} + c_1 P - k\dot{\theta} Q] dx \end{aligned}$$

By integrating by parts and keeping the boundary conditions (2.33) into account, we get

$$\begin{aligned} I_1 = \int_\Omega \Big[& \frac{1}{k^2}|\nabla\dot{f}|^2 + \dot{f}^2|\mathbf{A} - \nabla\theta|^2 + 2f\dot{f}(\mathbf{A} - \nabla\theta)\cdot(\dot{\mathbf{A}} - \nabla\dot{\theta}) + \frac{c_1}{k^2}\nabla f \cdot \nabla\dot{f} + c_1 f\dot{f}|\mathbf{A} - \nabla\theta|^2 \\ & -2\dot{f}\dot{\theta}\nabla f \cdot (\mathbf{A} - \nabla\theta) + f\dot{f}\phi\triangle\theta - f^2(\dot{\theta} - \phi)\triangle\dot{\theta} + 2f(\dot{\theta} - \phi)\nabla\dot{f} \cdot (\mathbf{A} - \nabla\theta) \\ & +2f(\dot{\theta} - \phi)\nabla f \cdot (\dot{\mathbf{A}} - \nabla\dot{\theta}) - c_1 f^2(\dot{\theta} - \phi)\triangle\theta + 2c_1 f(\dot{\theta} - \phi)\nabla f \cdot (\mathbf{A} - \nabla\theta) - f\dot{\theta}(\dot{\theta} - \phi)\triangle f \\ & +k^2 f^2(\dot{\theta} - \phi)\dot{\theta}|\mathbf{A} - \nabla\theta|^2 - 2f(\dot{\theta} - \phi)\nabla f \cdot \nabla\phi - f^2(\nabla\dot{\theta} - \nabla\phi)\cdot\nabla\phi \Big] dx \end{aligned}$$

Since $\nabla\cdot\mathbf{A} = 0$, in the previous expression we can replace $\triangle\theta$ and $\triangle\dot{\theta}$ by $-\nabla\cdot(\mathbf{A} - \nabla\theta)$ and $-\nabla\cdot(\dot{\mathbf{A}} - \nabla\dot{\theta})$ respectively and integrating by parts. A straightforward computation proves that I_1 can be written as

$$\begin{aligned} I_1 = \int_\Omega \Big[& |\mathbf{R}|^2 + |\mathbf{S}|^2 + \frac{c_1}{2}\frac{d}{dt}\left(\frac{1}{k^2}|\nabla f|^2 + f^2|\mathbf{A} - \nabla\theta|^2 \right) - \frac{k^2}{4}\phi^2|\nabla f|^2 - \frac{1}{4}f^2\phi^2|\mathbf{A} - \nabla\theta|^2 \\ & +f(\phi - \dot{\theta})\nabla f \cdot \nabla\phi - c_1 f^2(\dot{\mathbf{A}} - \nabla\phi)\cdot(\mathbf{A} - \nabla\theta) \\ & -f^2|\dot{\mathbf{A}}|^2 - f\dot{f}(\mathbf{A} - \nabla\theta)\cdot\nabla\phi + f\phi\nabla f \cdot \dot{\mathbf{A}} \Big] dx \end{aligned}$$

where

$$\begin{aligned} \mathbf{R} &= \dot{f}(\mathbf{A} - \nabla\theta) + f(\dot{\mathbf{A}} - \nabla\dot{\theta}) + f\nabla\phi - \dot{\theta}\nabla f + \frac{1}{2}\phi\nabla f \\ \mathbf{S} &= \frac{1}{k}\nabla\dot{f} + kf\dot{\theta}(\mathbf{A} - \nabla\theta) - \frac{k}{2}f\phi(\mathbf{A} - \nabla\theta) \end{aligned}$$

Concerning I_2, we observe that

$$I_2 = \int_\Omega \Big[-\frac{1}{k^2}(f^3-f)\triangle \dot f + \dot f (f^3-f)|\mathbf{A}-\nabla\theta|^2 + 2(f^4-f^2)(\mathbf{A}-\nabla\theta)\cdot(\dot{\mathbf{A}}-\nabla\dot\theta)$$
$$-\frac{c_1}{k^2}(f^3-f)\triangle f + c_1 f^2(f^2-1)|\mathbf{A}-\nabla\theta|^2 + (f^4-f^2)\dot\theta\triangle\theta - 2\dot\theta(f^3-f)\nabla f\cdot(\mathbf{A}-\nabla\theta)\Big]\, dx$$

and, by integrating by parts, we obtain

$$I_2 = \int_\Omega \Big[\frac{1}{k^2}(3f^2-1)\nabla f\cdot\nabla\dot f + (f^3-f)(\mathbf{A}-\nabla\theta)\cdot[\dot f(\mathbf{A}-\nabla\theta) + f(\dot{\mathbf{A}}-\nabla\dot\theta)]$$
$$+(f^4-f^2)\dot{\mathbf{A}}\cdot(\mathbf{A}-\nabla\theta) + 2\dot\theta(2f^3-f)\mathbf{A}\cdot\nabla f + \frac{c_1}{k^2}(3f^2-1)|\nabla f|^2 + c_1 f^2(f^2-1)|\mathbf{A}-\nabla\theta|^2$$
$$-2f\dot\theta(2f^2-1)\nabla\theta\cdot\nabla f - 2\dot\theta(f^3-f)\nabla f\cdot(\mathbf{A}-\nabla\theta)\Big]\, dx$$

The definition of $\mathbf{R}$ and $\mathbf{S}$, yields

$$I_2 = \int_\Omega \Big\{ (f^3-f)(\mathbf{A}-\nabla\theta)\cdot\mathbf{R} + \frac{1}{k}(3f^2-1)\nabla f\cdot\mathbf{S}$$
$$+f^3\phi(\mathbf{A}-\nabla\theta)\cdot\nabla f - (f^4-f^2)(\mathbf{A}-\nabla\theta)\cdot(\nabla\phi-\dot{\mathbf{A}}) + \frac{c_1}{k^2}(3f^2-1)|\nabla f|^2$$
$$+c_1 f^2(f^2-1)|\mathbf{A}-\nabla\theta|^2\Big\} dx$$

Let us consider the equation (2.31), multiply it by $\nabla\times\nabla\times\dot{\mathbf{A}} + c_2\nabla\times\nabla\times\mathbf{A}$ and integrate in Ω. Keeping the boundary conditions $(2.33)_2$ into account, we get the relation

$$\int_\Omega \frac{d}{dt}\left[\frac{1}{2}|\nabla\times\nabla\times\mathbf{A}|^2 + \frac{\eta c_2}{2}|\nabla\times\mathbf{A}|^2\right] dx$$
$$+\int_\Omega \Big[\eta|\nabla\times\dot{\mathbf{A}}|^2 + |\nabla\times\nabla\times\mathbf{A}|^2 + \nabla\times\dot{\mathbf{A}}\cdot[2f\nabla f\times(\mathbf{A}-\nabla\theta) + f^2\nabla\times\mathbf{A}]$$
$$+c_2 f^2(\mathbf{A}-\nabla\theta)\cdot\nabla\times\nabla\times\mathbf{A}\Big]\, dx + \int_{\partial\Omega}\dot{\mathbf{A}}\cdot\mathbf{H}_{ex}\times\mathbf{n}da = 0 \qquad (5.6)$$

From the relations (5.5)-(5.6), we obtain

$$\frac{d}{dt}\mathcal{E}_1 + \frac{d}{dt}\int_\Omega\left[\frac{c_1}{2k^2}|\nabla f|^2 + \frac{c_1}{2}|f(\mathbf{A}-\nabla\theta)|^2 + \frac{c_2\eta}{2}|\nabla\times\mathbf{A}|^2 + \frac{1}{4}(f^2-1)^2\right] dx$$
$$+\int_\Omega\left[c_1P^2 + c_1Q^2 + \frac{\eta}{k^2}(\triangle\phi)^2 + \eta|\nabla\times\dot{\mathbf{A}}|^2 + |\nabla\times\nabla\times\mathbf{A}|^2 + |\mathbf{R}|^2 + |\mathbf{S}|^2\right] dx \le I_3$$

where $\mathcal{E}_1$ is defined by (5.1) and

$$I_3 = \int_\Omega\Big[\frac{1}{4}\phi^2|\nabla f|^2 + \frac{k^2}{4}f^2\phi^2|\mathbf{A}-\nabla\theta|^2 - f(\phi-\dot\theta)\nabla f\cdot\nabla\phi + c_1 f^2(\dot{\mathbf{A}}-\nabla\phi)\cdot(\mathbf{A}-\nabla\theta)$$

$$+f^2|\dot{\mathbf{A}}|^2 + f\dot{f}(\mathbf{A}-\nabla\theta)\cdot\nabla\phi - f\phi\nabla f\cdot\dot{\mathbf{A}}\Big]\,dx$$

$$-\int_\Omega \Big\{(f^3-f)(\mathbf{A}-\nabla\theta)\cdot\mathbf{R} + \frac{1}{k}(3f^2-1)\nabla f\cdot\mathbf{S}$$

$$+f^3\phi(\mathbf{A}-\nabla\theta)\cdot\nabla f - (f^4-f^2)(\mathbf{A}-\nabla\theta)\cdot(\nabla\phi-\dot{\mathbf{A}}) + \frac{c_1}{k^2}(3f^2-1)|\nabla f|^2$$

$$+c_1 f^2(f^2-1)|\mathbf{A}-\nabla\theta|^2 + (f^2-1)f\dot{f}\Big\}dx - \int_{\partial\Omega}\dot{\mathbf{A}}\cdot\mathbf{H}_{ex}\times\mathbf{n}da$$

$$-\int_\Omega \Big\{\nabla\times\dot{\mathbf{A}}\cdot[2f\nabla f\times(\mathbf{A}-\nabla\theta) + f^2\nabla\times\mathbf{A}] + c_2 f^2(\mathbf{A}-\nabla\theta)\cdot\nabla\times\nabla\times\mathbf{A}\Big\}\,dx \qquad (5.7)$$

Hence

$$\frac{d}{dt}(\mathcal{E}_1+\gamma\mathcal{E}_0) + \int_\Omega \Big[c_1P^2 + c_1Q^2 + \frac{\eta}{k^2}(\triangle\phi)^2 + \eta|\nabla\times\dot{\mathbf{A}}|^2 + |\nabla\times\nabla\times\mathbf{A}|^2 + |\mathbf{R}|^2 + |\mathbf{S}|^2\Big]\,dx \le I_3 \qquad (5.8)$$

where $\gamma = \min\{c_1, c_2\eta, 1\}$.

In order to estimate the right-hand side of (5.8), we need some lemmas. We use repeatedly the Theorem 4.1 with $\Gamma' = 2\Gamma$. Moreover we denote by $C(\Gamma)$ a generic constant depending on Γ (i.e. depending on Ω and $\mathbf{H}_{ex}$), which may vary even in the same formula.

Lemma 5.1 *If the initial data satisfy the inequality $\mathcal{E}_0(f_0, \nabla\theta_0, \mathbf{A}_0) \le M$, then*

$$\|\nabla\phi\|_2 \le C(\Gamma) \qquad (5.9)$$

for $t > t_0$.

Proof. Consider the equation (2.34) and multiply it by ϕ. An integration by parts and use of the boundary conditions (2.33) yield

$$\int_\Omega \eta|\nabla\phi|^2dx \le -\int_\Omega f^2(\mathbf{A}-\nabla\theta)\cdot\nabla\phi dx$$

In view of Proposition 3.1, we have

$$\|\nabla\phi\|_2^2 \le \frac{1}{\eta}\int_\Omega |f^2(\mathbf{A}-\nabla\theta)\cdot\nabla\phi|dx \le \frac{1}{\eta}\|f(\mathbf{A}-\nabla\theta)\|_2\,\|\nabla\phi\|_2,$$

so that

$$\|\nabla\phi\|_2^2 \le \frac{1}{\eta^2}\|f(\mathbf{A}-\nabla\theta)\|_2^2 \le \frac{2}{\eta^2}\mathcal{E}_0(f,\nabla\theta,\mathbf{A})$$

The application of Theorem 4.1 proves (5.9). □

Lemma 5.2 *If $\Omega \subset \mathbb{R}^2$ and the initial data satisfy $\mathcal{E}_0(f_0, \nabla\theta_0, \mathbf{A}_0) \le M$, there exist positive constants $C_1(\Gamma), C_2(\Gamma)$ and $C_3(\Gamma)$ such that*

$$\begin{aligned} I_3 &\le C_1(\Gamma) + C_2(\Gamma)(\|P\|_2^2 + \|Q\|_2^2 + \|\nabla\times\nabla\times\mathbf{A}\|_2^2) + C_3(\Gamma)[\|kf(\dot\theta-\phi)\|_2^2 + \|\dot{\mathbf{A}}\|_2^2 + \|\dot f\|_2^2] \\ &+ \frac{1}{2}(\|\mathbf{R}\|_2^2 + \|\mathbf{S}\|_2^2 + \eta\|\nabla\times\dot{\mathbf{A}}\|_2^2 + \frac{\eta}{k^2}\|\triangle\phi\|_2^2) \end{aligned}$$

Proof. In view of the definitions (5.2), we have

$$P^2 + Q^2 = |(\frac{i}{k}\nabla + \mathbf{A})^2\psi|^2 = |-\frac{1}{k^2}\triangle\psi + \mathbf{A}^2\psi + \frac{2i}{k}\mathbf{A}\cdot\nabla\psi|^2$$

Therefore, by means of the inequality $|x+y|^2 \le 2|x|^2 + 2|y|^2$, $\forall x, y \in \mathbb{C}$, we obtain

$$|\triangle\psi|^2 \le 2k^4(P^2+Q^2) + 2k^4|\frac{2i}{k}\mathbf{A}\cdot\nabla\psi + \mathbf{A}^2\psi|^2 \le 2k^4(P^2+Q^2) + \frac{16}{k^2}|\mathbf{A}|^2|\nabla\psi|^2 + 4k^4|\mathbf{A}|^4|\psi|^2$$

The previous inequality and the condition $|\psi| \le 1$, yield

$$\|\triangle\psi\|_2^2 \le 2k^4(\|P\|_2^2 + \|Q\|_2^2) + \left(4k^4 + \frac{8}{\nu k^2}\right)\|\mathbf{A}\|_4^4 + \frac{8\nu}{k^2}\|\nabla\psi\|_4^4$$

for each $\nu > 0$. Moreover, when $\Omega \subset \mathbb{R}^2$, the classical interpolation inequality

$$\|h\|_4^2 \le K_3\|h\|_2\,\|h\|_{H^1} \qquad h \in H^1(\Omega) \tag{5.10}$$

implies

$$\|\nabla\psi\|_4^4 \le K_3^2\|\nabla\psi\|_2^2\|\nabla\psi\|_{H^1}^2 \le K_3^2\left(\|\nabla\psi\|_2^4 + \|\nabla\psi\|_2^2\|\triangle\psi\|_2^2\right),$$

so that, in view of Theorem 4.1, we obtain

$$\|\triangle\psi\|_2^2 \le 2k^4(\|P\|_2^2 + \|Q\|_2^2) + C(\Gamma) + 8\nu C(\Gamma)\|\triangle\psi\|_2^2$$

By choosing ν such that $8\nu C(\Gamma) < \frac{1}{2}$, we have

$$\|\triangle\psi\|_2^2 \le 4k^4(\|P\|_2^2 + \|Q\|_2^2) + C(\Gamma) \tag{5.11}$$

In order to estimate the terms of (5.7), we will use Holder's and Young's inequalities, Sobolev embeddings, the interpolation inequality (5.10), the relations (5.9), (5.11) and the condition $f^2 \le 1$. Accordingly, we have

$$\begin{aligned}
J_1 &:= \int_\Omega \phi^2(|\nabla f|^2 + k^2 f^2|\mathbf{A}-\nabla\theta|^2)dx \le \int_\Omega [\phi^2|\nabla\psi|^2 + 2k^2\phi^2 f^2(\mathbf{A}-\nabla\theta)\cdot\mathbf{A} - k^2\phi^2 f^2|\mathbf{A}|^2]dx \\
&\le \|\phi\|_4^2\|\nabla\psi\|_4^2 + 2k^2\|f(\mathbf{A}-\nabla\theta)\|_2\,\|\phi\|_6^2\|\mathbf{A}\|_6 + k^2\|\mathbf{A}\|_4^2\,\|\phi\|_4^2 \le K_3\|\phi\|_4^2\|\nabla\psi\|_2\|\nabla\psi\|_{H^1} + C(\Gamma) \\
&\le C(\Gamma) + \|\triangle\psi\|_2^2 \le C(\Gamma) + 4k^2(\|P\|_2^2 + \|Q\|_2^2)
\end{aligned}$$

$$\begin{aligned}
J_2 &:= \int_\Omega f(\phi-\dot\theta)\nabla f\cdot\nabla\phi \le \|f(\phi-\dot\theta)\|_2\,\|\nabla\psi\|_4\|\nabla\phi\|_4 \\
&\le \|f(\phi-\dot\theta)\|_2^2 + K_3^2\|\nabla\psi\|_2\,\|\nabla\phi\|_2\,\|\nabla\psi\|_{H^1}\,\|\nabla\phi\|_{H^1} \\
&\le \|f(\phi-\dot\theta)\|_2^2 + C(\Gamma)\|\triangle\psi\|_2^2 + \frac{\eta}{4k^2}\|\triangle\phi\|_2^2 + C(\Gamma)
\end{aligned}$$

$$J_3 := \int_\Omega [c_1 f^2(\dot{\mathbf{A}}-\nabla\phi)\cdot(\mathbf{A}-\nabla\theta) + f^2|\dot{\mathbf{A}}|^2]dx \le c_1\|f(\mathbf{A}-\nabla\theta)\|_2(\|\dot{\mathbf{A}}\|_2 + \|\nabla\phi\|_2) + \|\dot{\mathbf{A}}\|_2^2$$

$$
\begin{aligned}
&\leq \quad C(\Gamma)+2\|\dot{\mathbf{A}}\|_2^2 \\
J_4 \quad &:= \quad \int_\Omega [f\dot{f}(\mathbf{A}-\nabla\theta)\cdot\nabla\phi - f\phi\nabla f\cdot\dot{\mathbf{A}}]dx \leq \|\dot{f}\|_2\,\|\nabla\phi\|_4(\|f\mathbf{A}\|_4+\|f\nabla\theta\|_4)+\|\dot{\mathbf{A}}\|_2\,\|\nabla f\|_4\|\phi\|_4 \\
&\leq \quad \|\dot{f}\|_2\,\|\nabla\phi\|_4(\|\mathbf{A}\|_4+\frac{1}{k}\|\nabla\psi\|_4)+\|\dot{\mathbf{A}}\|_2\,\|\nabla\psi\|_4\|\phi\|_4 \\
&\leq \quad C(\Gamma)(\|\dot{f}\|_2^2+\|\dot{\mathbf{A}}\|_2^2)+K_3\|\triangle\phi\|_2\,\|\nabla\phi\|_2+K_3^2\|\nabla\phi\|_2\,\|\nabla\psi\|_2\,\|\triangle\phi\|_2\,\|\triangle\psi\|_2+K_3\|\triangle\psi\|_2\,\|\nabla\psi\|_2 \\
&\leq \quad C(\Gamma)(\|\dot{f}\|_2^2+\|\dot{\mathbf{A}}\|_2^2)+C(\Gamma)(\|P\|_2^2+\|Q\|_2^2)+\frac{\eta}{4k^2}\|\triangle\phi\|_2^2+C(\Gamma)
\end{aligned}
$$

$$
\begin{aligned}
J_5 \quad &:= \quad \int_\Omega [-f(f^2-1)(\mathbf{A}-\nabla\theta)\cdot\mathbf{R}-\frac{1}{k}(3f^2-1)\nabla f\cdot\mathbf{S}]dx \\
&\leq \quad 2\|f(\mathbf{A}-\nabla\theta)\|_2\,\|\mathbf{R}\|_2+\frac{4}{k}\|\nabla f\|_2\,\|\mathbf{S}\|_2 \leq \frac{1}{2}\|\mathbf{R}\|_2^2+\frac{1}{2}\|\mathbf{S}\|_2^2+C(\Gamma)
\end{aligned}
$$

$$
\begin{aligned}
J_6 \quad &:= \quad \int_\Omega f^3\phi(\mathbf{A}-\nabla\theta)\cdot\nabla f dx \leq \|f(\mathbf{A}-\nabla\theta)\|_2\,\|\phi\|_4\|\nabla f\|_4 \leq C(\Gamma)+K_3\|\nabla\psi\|_2\,\|\nabla\psi\|_{H^1} \\
&\leq \quad C(\Gamma)+\|\triangle\psi\|_2^2 \leq C(\Gamma)+4k^2(\|P\|_2^2+\|Q\|_2^2)
\end{aligned}
$$

$$
\begin{aligned}
J_7 \quad &:= \quad \int_\Omega [-(f^4-f^2)(\mathbf{A}-\nabla\theta)\cdot(\nabla\phi-\dot{\mathbf{A}})+\frac{c_1}{k^2}(3f^2-1)|\nabla f|^2+c_1f^2(f^2-1)|\mathbf{A}-\nabla\theta|^2 \\
&\quad\; +(f^2-1)f\dot{f}]dx \leq C(\Gamma)+\|\dot{\mathbf{A}}\|_2^2+\|\dot{f}\|_2^2
\end{aligned}
$$

$$
\begin{aligned}
J_8 \quad &:= \quad -\int_{\partial\Omega} \dot{\mathbf{A}}\cdot\mathbf{H}_{ex}\times\mathbf{n}da \leq K_2\|\nabla\times\dot{\mathbf{A}}\|_2\,\|\mathbf{H}_{ex}\times\mathbf{n}\|_{H^{-1/2}(\partial\Omega)} \\
&\leq \quad \frac{\eta}{4}\|\nabla\times\dot{\mathbf{A}}\|_2^2+\frac{K_2^2}{\eta}\|\mathbf{H}_{ex}\times\mathbf{n}\|_{H^{-1/2}(\partial\Omega)}^2
\end{aligned}
$$

$$
\begin{aligned}
J_9 \quad &:= \quad -\int_\Omega \nabla\times\dot{\mathbf{A}}\cdot[2f\nabla f\times(\mathbf{A}-\nabla\theta)+f^2\nabla\times\mathbf{A}]dx \\
&\leq \quad 2\|\nabla\times\dot{\mathbf{A}}\|_2\,\|\nabla f\|_4\,\|f(\mathbf{A}-\nabla\theta)\|_4+\|\nabla\times\dot{\mathbf{A}}\|_2\,\|\nabla\times\mathbf{A}\|_2 \\
&\leq \quad \frac{\eta}{4}\|\nabla\times\dot{\mathbf{A}}\|_2^2+\frac{6}{\eta}\|\nabla f\|_4^2(\|f\mathbf{A}\|_4+\|f\nabla\theta\|_4)^2+\frac{3}{\eta}\|\nabla\times\mathbf{A}\|_2^2 \\
&\leq \quad \frac{\eta}{4}\|\nabla\times\dot{\mathbf{A}}\|_2^2+\frac{12}{\eta}\|\nabla f\|_4^2(\|\mathbf{A}\|_4^2+\|f\nabla\theta\|_4^2)+C(\Gamma) \\
&\leq \quad \frac{\eta}{4}\|\nabla\times\dot{\mathbf{A}}\|_2^2+C(\Gamma)\|\nabla\psi\|_4^2+\frac{12K_3^2}{\eta k^2}\|\nabla\psi\|_2^2\|\triangle\psi\|_2^2+C(\Gamma) \\
&\leq \quad \frac{\eta}{4}\|\nabla\times\dot{\mathbf{A}}\|_2^2+C(\Gamma)\|\triangle\psi\|_2^2+C(\Gamma) \\
J_{10} \quad &:= \quad -\int_\Omega c_2f^2(\mathbf{A}-\nabla\theta)\cdot\nabla\times\nabla\times\mathbf{A}dx \leq C(\Gamma)+\|\nabla\times\nabla\times\mathbf{A}\|_2^2
\end{aligned}
$$

From the previous estimates, we deduce the inequality

$$\begin{aligned} I_3 &\le C_1(\Gamma) + C_2(\Gamma)(\|P\|_2^2 + \|Q\|_2^2 + \|\nabla\times\nabla\times\mathbf{A}\|_2^2) + C_3(\Gamma)[\|kf(\dot\theta-\phi)\|_2^2 + \|\dot{\mathbf{A}}\|_2^2 + \|\dot f\|_2^2] \\ &+ \frac{1}{2}(\|\mathbf{R}\|_2^2 + \|\mathbf{S}\|_2^2 + \eta\|\nabla\times\dot{\mathbf{A}}\|_2^2 + \frac{\eta}{k^2}\|\triangle\phi\|_2^2) \end{aligned}$$

□

Lemma 5.3 *If $\mathcal{E}_0(f_0, \nabla\theta_0, \mathbf{A}_0) \le M$, the following inequalities hold*

$$\int_t^{t+1} \left[\|\dot f\|_2^2 + \|kf(\dot\theta-\phi)\|_2^2 + \eta\|\mathbf{A}\|_2^2\right] d\tau \le C(\Gamma) \qquad (5.12)$$

$$\int_t^{t+1} \mathcal{E}_1(\tau) d\tau \le C(\Gamma) \qquad (5.13)$$

for $t > t_0$.

Proof. Let us consider the relation (4.9) and integrate in the time interval $[t, t+1]$. We have

$$\mathcal{F}(t+1) - \mathcal{F}(t) + \lambda\int_t^{t+1}\mathcal{F}(\tau)d\tau + \int_t^{t+1}\left[\|\dot f\|_2^2 + \|kf(\dot\theta-\phi)\|_2^2 + \eta\|\mathbf{A}\|_2^2\right] d\tau \le C(\Gamma).$$

Since the functional $\mathcal{F}$ is positive definite, we obtain

$$\int_t^{t+1}\left[\|\dot f\|_2^2 + \|kf(\dot\theta-\phi)\|_2^2 + \eta\|\mathbf{A}\|_2^2\right] d\tau \le C(\Gamma) + \mathcal{F}(t).$$

Thus, keeping (4.7) into account, by Theorem 4.1, we prove (5.12).

In order to prove (5.13) we observe that, by definition (5.1), we obtain

$$\int_t^{t+1}\mathcal{E}_1(\tau)d\tau = \frac{1}{2}\int_t^{t+1}\int_\Omega \left[P^2 + Q^2 + |\nabla\times\nabla\times\mathbf{A}|^2\right] dx\, d\tau\,.$$

Moreover, by using equations (2.30)-(2.32), we have

$$\begin{aligned} &\int_t^{t+1}\mathcal{E}_1(\tau)d\tau = \frac{1}{2}\int_t^{t+1}\int_\Omega \left\{[\dot f + f(f^2-1)]^2 + k^2f^2(\dot\theta-\phi)^2 + [\eta(\dot{\mathbf{A}}-\nabla\phi) + f^2(\mathbf{A}-\nabla\theta)]^2\right\} dxd\tau \\ &\le \int_t^{t+1}\left[\|\dot f\|_2^2 + \|f(f^2-1)\|_2^2 + \frac{k^2}{2}\|f(\dot\theta-\phi)\|_2^2 + 2\eta^2(\|\dot{\mathbf{A}}\|_2^2 + \|\nabla\phi\|_2^2) + \|f^2(\mathbf{A}-\nabla\theta)\|_2^2\right] d\tau \end{aligned}$$

so that in view of (5.12), of Lemma 5.1 and of Theorem 4.1, we obtain (5.13). □

Theorem 5.1 *If $\Omega \subset \mathbb{R}^2$, the system (2.30)-(2.32) possesses a maximal attractor.*

Proof. In view of Lemma 5.2, from (5.8) we have

$$\frac{d}{dt}(\mathcal{E}_1+\gamma\mathcal{E}_0)+\int_\Omega\left[c_1P^2+c_1Q^2+\frac{\eta}{2k^2}(\triangle\phi)^2+\frac{\eta}{2}|\nabla\times\dot{\mathbf{A}}|^2+|\nabla\times\nabla\times\mathbf{A}|^2+\frac{1}{2}|\mathbf{R}|^2+\frac{1}{2}|\mathbf{S}|^2\right]dx$$
$$\leq C_1(\Gamma)+2C_2(\Gamma)\mathcal{E}_1+C_3(\Gamma)[\|kf(\dot{\theta}-\phi)\|_2^2+\|\dot{\mathbf{A}}\|_2^2+\|\dot{f}\|_2^2]$$

Hence

$$\frac{d}{dt}(\mathcal{E}_1+\gamma\mathcal{E}_0)\leq C_1(\Gamma)+2C_2(\Gamma)(\mathcal{E}_1+\gamma\mathcal{E}_0)+C_3(\Gamma)[\|kf(\dot{\theta}-\phi)\|_2^2+\|\dot{\mathbf{A}}\|_2^2+\|\dot{f}\|_2^2]$$

The inequalities (5.12) and (5.13) allow to apply the uniform Gronwall lemma (see [13]) which proves that $\mathcal{E}_1(t)$ is bounded for $t>t_0$. This guarantees the existence of the maximal attractor for the system. □

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