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with Vertical Linkages**

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Abstract

I propose a mechanism for the coexistence of vertical integration and outsourcing within an industry. In a monopolistically competitive model with incomplete contracts, final-good producers under either organizational form combine labor and differentiated inputs to produce fixed capital. Since its unit cost depends on the industry price index, distinct zero-profit conditions can hold, sustaining a mixed free-entry equilibrium. The forms differ in fixed-capital cost per unit of contribution to the price index; this ranking determines comparative statics and whether local stability is possible. I derive a sufficient condition ensuring stability whenever the mixed equilibrium exists. In numerical parameterizations using empirically standard elasticities of substitution, the condition holds throughout the admissible range of vertical-linkage intensity. Within the stable region, larger markets raise the integrated share without changing output prices or supplier bargaining terms. Relative firm size depends on governance costs and upstream bargaining share, so the model imposes no uniform ranking.

Keywords: Vertical integration; Outsourcing; Vertical linkages; Industry equilibrium; Stability of equilibrium; Capital flow table.

JEL Classification: C62, D23, D43, F12, L23.

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Non-technical summary

Why do firms that make the same product organize production so differently? In many industries some producers make their key components in-house while their direct competitors buy them from independent suppliers, and both arrangements persist side by side. Standard theory finds this hard to explain: identical firms should all make the same choice, so either everyone integrates or everyone outsources. Coexistence is usually obtained by assuming that firms differ from the start, typically in productivity; the evidence that integrated firms are systematically more productive is, however, mixed.

This paper proposes an explanation that does not rely on differences across firms. Its starting point is a fact documented with the U.S. Capital Flow Table: the machinery, tooling and equipment a manufacturer needs to set up production are themselves made by other manufacturers, often in the same or in closely related industries. In the equipment-producing sector of U.S. manufacturing, about 62 percent of investment in equipment and software consists of goods produced within that same group of industries. This creates a feedback loop. When more firms are active, the range of capital goods on offer is wider and their effective cost lower, so setting up a new firm becomes cheaper. This loop is what allows the two arrangements to coexist, because they are exposed to it to different degrees: the setup cost of an integrated firm consists entirely of such capital goods, whereas an outsourcing arrangement also involves a supplier whose overhead is mostly labor. Thanks to this difference, both can just cover their costs at the same time, with nothing forcing either out.

The paper then asks when a mixed industry is stable, that is, whether it survives small disturbances. Two conditions are needed. The first concerns setup costs, measured not in absolute terms but relative to how much each entrant expands and cheapens the supply of capital goods for everyone: on this measure the two arrangements must be close, with integration coming out lighter, which is compatible with integrated firms having the larger setup cost in money terms. The second is that the market must not be so large, nor the loop so strong, that the cost reduction brought by entry overwhelms the discipline of competition; at the degrees of product substitutability standard in empirical trade studies, this condition is always satisfied.

Two implications stand out. Within a stable mixed industry, a larger market raises the share of integrated firms without changing prices or the terms on which suppliers are paid: the whole adjustment runs through the industry's composition. Neither arrangement is uniformly larger: whether integrated firms are bigger than their outsourcing rivals depends on the governance costs of integration and on suppliers' bargaining strength.

1 Introduction

The simultaneous coexistence of different organizational forms in a given industry, and specifically of vertically integrated and non-integrated firms, is a well-documented empirical fact, even within narrowly defined markets (e.g. [Forbes and Lederman, 2010](#), [Hortaçsu and Syverson, 2007](#), [Lin et al., 2020](#)). Consequently, developing models that feature the endogenous emergence of organizational heterogeneity has been a significant theoretical challenge. This paper presents a model in which firms’ boundaries are shaped by vertical linkages in the creation of fixed capital.

The mechanism rests on the observation that fixed-capital production draws on differentiated capital goods supplied by other firms in the same or in closely related industries. The 1997 BEA Capital Flow Table ([U.S. BEA, 2003](#)) shows that 61.7 percent of the equipment and software investment undertaken by the equipment complex (NAICS 333–336: machinery; computer and electronic products; electrical equipment; and transportation equipment) consisted of capital goods produced within the complex. At the three-digit level, the within-industry share ranged from 2.4 percent in electrical equipment to 44.4 percent in machinery. Even at the four-digit level, metalworking machinery sourced 30.2 percent of its equipment and software investment from its own industry, and computer and peripheral equipment 13.9 percent. The equipment complex also carries substantial weight in the economy: its \$64.8 billion of equipment and software investment was 8.1 percent of the total recorded for the 123 using industries and 40 percent of the total for the 53 manufacturing industries.¹

[Grossman and Helpman \(2002\)](#) find that, generically, no industry will have both vertically integrated and specialized producers coexisting in equilibrium.² The setting of [Antràs \(2003\)](#) similarly delivers either pervasive integration or pervasive outsourcing, except when knife-edge conditions hold. In that model the organizational choice is a decision about the allocation of residual rights of control, following [Grossman and Hart \(1986\)](#). In [McLaren \(2000\)](#), vertical integration thins the market for independent inputs, thereby worsening the hold-up problem faced by non-integrated firms and generating strategic complementarity in organizational choices. With ex ante identical firms, complete integration is the unique pure-strategy equilibrium in a sufficiently small closed economy, whereas a sufficiently large economy admits two pure-strategy equilibria: complete integration and universal reliance on independent suppliers.³ Within-industry coexistence of the two procurement modes arises in his framework only under ex ante heterogeneity in the suppliers’ aptitude for flexibility: the most flexible suppliers remain independent, while the remaining buyer-supplier pairs integrate. Thus, with ex ante identical firms these mechanisms deliver a single

¹Appendix [A.1](#) details the construction of these measures and reports them for each detailed industry in the complex.

²[Grossman and Helpman \(2002\)](#), Proposition 1.

³[McLaren \(2000\)](#), Proposition 6.

organizational form industry-wide; coexistence, where it emerges, requires exogenous heterogeneity across firms.

Legros and Newman (2013) show that vertically integrated and non-integrated firms can co-exist in competitive equilibrium even when enterprises are ex ante technologically identical. The mechanism rests on the output differential between organizational forms: integration improves coordination and therefore generates greater expected output than non-integration. At the price at which managers are indifferent between the two ownership structures, product-market clearing may require positive shares of enterprises to adopt each form, and such mixed equilibria arise for a generic set of demand functions.⁴ Productivity heterogeneity introduces an organizational sorting margin: more productive enterprises face lower price thresholds for integration and are therefore more likely to be integrated.⁵ Dam and Serfes (2021) develop this margin systematically by introducing one-sided productivity heterogeneity among suppliers and endogenizing the division of surplus through matching. Under fairly general conditions, the equilibrium ownership structure is monotone: at any given product-market price, enterprises above a unique productivity cutoff integrate, whereas those below it remain non-integrated; the enterprise at the cutoff is indifferent between the two organizational forms. Conconi et al. (2012) incorporate the ownership-choice mechanism of Legros and Newman (2013) into a two-country trade model and study how liberalization reshapes firm boundaries. They show that both the liberalization of product markets, which moves world prices across a fixed organizational cutoff, and the liberalization of factor markets, which moves the cutoff itself, can trigger restructuring within countries. The ex ante heterogeneity introduced in their model plays a different role from that in the organizational-sorting literature. Upstream suppliers differ only in their productivity when operating independently in the numeraire sector, while they are equally productive when matched with a downstream supplier. This heterogeneity therefore determines selection between matched production and stand-alone production of the numeraire good and, through this selection, the common equilibrium payoff earned by matched upstream suppliers. This payoff determines a country-specific organizational price cutoff that is common across sectors. Thus, heterogeneity shifts the organizational cutoff faced by enterprises rather than generating within-sector sorting by enterprise productivity. Their main comparative statics accordingly take the form of sector-wide waves of mergers or divestitures, with sectors switching between full integration and full non-integration rather than adjusting the integrated share smoothly. Coexistence of different organizational forms can arise only when the equilibrium product price lies exactly at the cutoff, a configuration that their analysis sets aside as a boundary case.⁶

⁴Legros and Newman (2013), Proposition 2.

⁵Legros and Newman (2013), Section III.B.

⁶Conconi et al. (2012), footnotes 17 and 19.

Relative to binary-choice models, in which *ex ante* identical enterprises can adopt different organizational forms only at the product-market price that leaves them indifferent, [Alfaro et al. \(2016\)](#) formulate vertical integration as an intensive-margin choice. An enterprise chooses a degree of integration, interpreted as the number of complementary inputs produced within the firm. Because integration raises productivity but entails a cost that is independent of the output price, the optimal degree of integration increases with the product-market price. This formulation allows firm boundaries to vary along an intensive margin rather than through discrete switches between integration and non-integration. By itself, however, it does not generate within-industry variation in integration among otherwise identical enterprises facing the same price, provided that their optimal choice is unique. When exogenous productivity differences are introduced, profits exhibit increasing differences in productivity and integration, and the model then yields the sorting prediction that more productive enterprises choose higher degrees of vertical integration.

The evidence reviewed above motivates the search for mechanisms that can sustain the within-industry coexistence of integrated and non-integrated firms without requiring either an output advantage of integration ([Conconi et al., 2012](#), [Legros and Newman, 2013](#)) or productivity differences across firms ([Alfaro et al., 2016](#), [Dam and Serfes, 2021](#)). This paper develops one such mechanism, based on vertical linkages in fixed-capital production. I study the implications of assuming that all firms producing the horizontally differentiated final goods (under either vertical integration or outsourcing) need, in addition to variable inputs, a Cobb–Douglas composite of the differentiated goods and labor to create the fixed capital needed for production. The existence of these vertical linkages has two primary effects: it creates an augmented demand schedule for differentiated varieties, from both consumers and producers; it adjusts the fixed cost component of the profits as a function of the price index of the horizontally differentiated varieties. These two effects allow, under some conditions laid down in the paper, for coexistence of both integrated and non-integrated firms in the same market. Positing that both integrated and specialized manufacturers need a basket of differentiated varieties, in addition to labor, to cover the fixed costs of production follows a well-established tradition in the economic geography literature ([Krugman and Venables, 1995](#), [Venables, 1996](#), [Ottaviano and Thisse, 2004](#), [Ottaviano and Robert-Nicoud, 2006](#)). Since the fixed costs affect the fulfillment of the zero profits condition under both production modes, by making them depend on the price index of the differentiated varieties through vertical linkages, I add an additional degree of freedom that enables a positive mass of both types of firms to coexist at equilibrium. In other words, while in [Grossman and Helpman \(2002\)](#) entry by one type of firm in numbers that ensure zero profits guarantees losses for the other type, this is not necessarily true in a framework where fixed capital is a composite intermediate input, similarly to [Krugman and Venables \(1995\)](#).

The local stability of the mixed equilibrium hinges on two margins. First, the fixed-capital costs per unit of contribution to the industry price index of the two organizational forms must be similar enough that neither dominates: when they differ too much, the equilibrium is either a saddle or fails to exist. Second, even when the former condition is met, stability further requires that market size and the strength of vertical linkages remain below thresholds at which the price-index feedback through which entry lowers the cost of fixed capital overturns product-market competition.

This paper contributes to two main strands of literature. First, it contributes to the theoretical literature, reviewed above, that explains within-industry heterogeneity in organizational form. Legros and Newman (2014) characterize the mechanism in Legros and Newman (2013) as demand-driven: at intermediate levels of demand, the equilibrium price is pinned at the level at which managers are indifferent between organizational forms, while market clearing determines industry composition, with greater demand increasing the share of integrated enterprises. Relative to this literature, the present paper shows that coexistence of organizational forms can arise in a monopolistic competition model with CES demand without appealing to systematic differences in output or productivity. At the same time, the model preserves one of the qualitative implications emphasized by this literature. In the locally stable mixed equilibrium, which arises in the region where integration has the lower fixed-capital cost per unit of contribution to the price index, larger markets are associated with a higher mass of integrated firms. The mechanism, however, is different. In Legros and Newman (2013) the mixed region is sustained by managerial indifference at a critical market price: at that price the supply correspondence is flat, and market clearing determines the share of integrated enterprises, so that a growing market raises the integrated share until the price leaves the indifference point. Here the mixed region is sustained by the strict zero-profit conditions of two freely entering populations, which are mutually compatible because the cost of fixed capital depends on the price index through vertical linkages. These conditions pin down the price index and the unit cost of fixed capital independently of market size; a larger market is therefore absorbed entirely by the composition of entrants, which shifts toward the organizational form with the lower fixed-capital burden.

Second, the paper speaks to the empirical literature on organizational choice and firm performance. The coexistence of vertically integrated and non-integrated firms is well documented, including within narrowly defined industries. A prominent class of explanations relies on within-industry productivity heterogeneity: more productive firms select into integration, while less productive firms outsource. The evidence on the productivity-integration relationship is mixed, however. In broad cross-country firm data, Alfaro et al. (2016, Appendix B) report a positive average association between their vertical-integration index and sales per worker, and integrated airlines achieve systematically better on-time performance (Forbes and Lederman, 2010). By contrast, the

productivity advantages of vertically integrated producers in cement and ready-mixed concrete shrink and lose statistical significance once total sales in the local ready-mixed concrete market are controlled for (Hortaçsu and Syverson, 2007), while the relationship between performance and organizational form in lodging is nonmonotonic (Lin et al., 2020). This mixed pattern motivates mechanisms that can sustain coexistence without imposing a uniform productivity ranking across organizational forms. The mechanism explored in this paper is complementary and operates in general equilibrium without invoking productivity differences. In what follows, I derive predictions that empirically discriminate it from both price-theoretic and heterogeneity-based alternatives.

The paper is organized as follows. Section 2 presents the model. Section 3 establishes the existence of a mixed equilibrium with integrated and non-integrated firms and characterizes the parameter restrictions required for coexistence. Section 4 studies the comparative statics. Section 5 analyzes the local stability of the mixed equilibrium. Section 6 examines welfare. Section 7 discusses robustness and derives empirical predictions that discriminate the mechanism from alternatives, and Section 8 concludes.

2 Model

The economy features L consumers with Cobb–Douglas preferences over a CES composite of differentiated varieties D and a perfectly competitive agricultural good Q ; labor is the numeraire (wage is normalized to 1). Sector D is monopolistically competitive with increasing returns, and firms can be vertically integrated, upstream specialists, or downstream specialists. I nest Grossman and Helpman (2002) in a framework with vertical linkages and adopt simplifying assumptions common in economic geography. Unlike Venables (1996) I do not separate upstream and downstream sectors, but following Krugman and Venables (1995) the horizontally differentiated composite serves both as the intermediate input and the consumption good. In line with Ottaviano and Thisse (2004) and Ottaviano and Robert-Nicoud (2006) the composite intermediate is required only for producing fixed capital, not as a variable input. Finally, I abstract from the search and matching frictions in Grossman and Helpman (2002) and assume a frictionless matching market. This simplification separates the role of vertical linkages from that of matching frictions: absent vertical linkages, coexistence is non-generic both in their frictional setting and, as the limiting case $\xi = 0$ with $\mu > 0$ in Section 7.1 shows, in the frictionless framework studied here.

Integrated producers face a variable cost of $\lambda > 1$ units of labor per unit of output, where the wedge above unity captures the governance costs inherent to vertical integration. Under outsourcing, by contrast, one unit of labor yields one unit of the intermediate component, which in

turn yields one unit of final output: the variable cost is normalized to unity. However, outsourcing operates under totally incomplete contracts (Antràs, 2016): third parties cannot verify component quality, so no enforceable quality provision can be written. Instead, the upstream supplier and the downstream buyer split surplus via Nash bargaining, which awards the supplier a revenue share ω together with a lump-sum transfer to the buyer that drives the supplier's profits to zero.

Fixed requirements differ by organizational role. Final-good producers (whether integrated or downstream specialists) must pay fixed costs f_v or f_s , respectively, produced by a Cobb–Douglas composite of labor and the same CES aggregate that enters the consumer's utility function. Upstream specialists, who supply only the intermediate component, instead pay a fixed cost f_m denominated entirely in labor. This asymmetry captures a natural difference in the complexity of overhead across stages of production. Setting up a final-good operation requires not only labor but also a diverse array of differentiated manufactured inputs: specialized machinery, customized tooling, testing and quality-control equipment, and packaging lines, all of which are varieties produced within the differentiated sector. Embedding the CES aggregate in the fixed costs of final-good producers reflects this richer input structure. Upstream specialists, by contrast, perform a narrower task: producing an intermediate component. Their setup requirements (basic equipment, workspace, workforce training) are plausibly intensive in homogeneous labor rather than in differentiated manufactures. Because the fixed costs of final-good producers draw on the CES aggregate, their entry feeds demand back into the varieties market: total demand for each differentiated variety is the sum of a consumer component and a fixed-input component, both of which depend on the price index P of the differentiated sector. Upstream specialists' labor-only fixed costs, by contrast, do not generate this feedback. The organizational form governs the strength of the vertical demand linkage: outsourcing introduces a labor-only overhead component (f_m) that integrated production does not, weakening the aggregate variety demand generated by fixed costs.

Timing is as follows: (1) firms enter as vertically integrated producers or as downstream specialists; (2) downstream specialists match with upstream partners from a sufficiently large pool to ensure a match; (3) all firms hire labor and purchase differentiated varieties to meet variable and fixed requirements, and production occurs. Consumers own firms, so profits are distributed to them.

I focus on a free-entry equilibrium in which net profits are zero. The goal is to characterize a mixed industrial structure where vertically integrated and disintegrated firms coexist: the masses of each type and the aggregate price index adjust jointly so that goods markets clear and the zero-profit conditions hold for both organizational forms. What follows describes the model in detail.

2.1 Consumption

The preferences of a representative consumer are Cobb–Douglas:

$$U = C_\xi D^\xi Q^{1-\xi}, \quad (1)$$

where C_ξ is a positive constant such that $C_\xi^{-1} = (\xi)^\xi (1-\xi)^{1-\xi}$, with $\xi > 0$, D is a CES aggregate of a mass equal to N of differentiated varieties,

$$D \equiv \left[\int_0^N y_i^\alpha di \right]^{1/\alpha},$$

$\alpha \in (0, 1)$ is the substitutability parameter, y_i the consumption of each variety, Q is the amount of the agricultural commodity. Calling I consumer's income, the utility function is maximized subject to income not exceeding I . Maximization yields that the consumer's demand function for a particular variety is

$$y_i = A p_i^{-1/(1-\alpha)}$$

and

$$A \equiv \frac{\xi I}{\int_0^N p_i^{-\alpha/(1-\alpha)} di} = \frac{\xi I}{P},$$

where p_i is a variety's price and the index P is defined as

$$P \equiv \int_0^N p_i^{-\alpha/(1-\alpha)} di,$$

being inversely related to each p_i .

2.2 Production

The agricultural good Q uses only labor with unit requirement γ : with wage normalized to 1, $p_Q = \gamma$. Sector D follows the setup above; here I describe the fixed-input technology.

I assume that the fixed requirement for vertically integrated firms, f_v , differs from the fixed requirement for disintegrated downstream producers, f_s , with no restriction on their relative magnitudes; upstream specialists in an outsourcing arrangement instead pay a labor-only fixed cost f_m . For final-good producers (whether integrated or disintegrated downstream), the fixed input is produced with a Cobb–Douglas technology exhibiting constant returns to scale:

$$f_z = K^\xi l_f^{1-\xi} \quad (2)$$

where $z \in \{v, s\}$ indexes the mode of organization, and K corresponds to the same CES aggregate

that enters utility,

$$K = \left[\int_0^N b_i^\alpha \mathrm{d}i \right]^{1/\alpha},$$

with b_i the quantity of variety i used for business purposes, while l_f is labor. The assumption that K is the same aggregate as in sector D follows [Krugman and Venables \(1995\)](#). Another assumption previously used in the literature is the equality of the Cobb–Douglas share parameter ξ across (1) and (2).⁷

Each final-good firm solves

$$\begin{cases} \min_{b_i, l_f} & \int_0^N p_i b_i \mathrm{d}i + l_f \\ \text{s.t.} & K^\xi l_f^{1-\xi} = f_z \end{cases}$$

which yields a total fixed-input cost equal to $f_z C_\xi P^{-\xi(1-\alpha)/\alpha}$. I write this more compactly as $f_z \kappa$, where $\kappa \equiv C_\xi P^{-\xi(1-\alpha)/\alpha}$ is the cost of one unit of fixed capital. The business demand for variety i by a vertically integrated producer, $b_{i,v}$, and by a downstream specialist, $b_{i,s}$, is therefore

$$b_{i,v} = \frac{\xi \kappa f_v}{\int_0^N p_j^{-\alpha/(1-\alpha)} \mathrm{d}j} p_i^{-1/(1-\alpha)}, \quad b_{i,s} = \frac{\xi \kappa f_s}{\int_0^N p_j^{-\alpha/(1-\alpha)} \mathrm{d}j} p_i^{-1/(1-\alpha)}.$$

Summing consumer and fixed-input components, total demand for variety i , denoted by x_i , is

$$x_i = y_i + s b_{i,s} + v b_{i,v} = (A + B) p_i^{-1/(1-\alpha)}, \quad (3)$$

where

$$B \equiv \frac{\xi \kappa (s f_s + v f_v)}{P}.$$

In what follows, I use $\mathcal{AB}$ as shorthand for the demand shifter $A + B$, namely

$$\mathcal{AB} \equiv A + B = \frac{\xi I + \xi \kappa (s f_s + v f_v)}{P}.$$

2.2.1 Profits under vertical integration

The price charged by a vertically integrated firm for the final good is

$$p_v = \frac{\lambda}{\alpha}.$$

The resulting sales are

$$x_v = \mathcal{AB} \left(\frac{\alpha}{\lambda} \right)^{1/(1-\alpha)}.$$

⁷[Ottaviano and Thisse \(2004\)](#); [Ottaviano and Robert-Nicoud \(2006\)](#). The implications of relaxing this assumption are shown in Section 7.1.

The total cost function under vertical integration is

$$TC_v = f_v \kappa + \lambda x_v.$$

Total profits of a vertically integrated firm are equal to total revenues minus total cost:

$$\Pi_v = \frac{(1 - \alpha)\lambda}{\alpha} x_v - f_v \kappa,$$

which can also be written as

$$\Pi_v = \mathcal{AB}(1 - \alpha) \left(\frac{\alpha}{\lambda} \right)^{\alpha/(1-\alpha)} - f_v \kappa. \quad (4)$$

2.2.2 Profits under outsourcing

The business environment under outsourcing is based on what [Antràs \(2016\)](#), p. 101, describes as the *totally incomplete* contracts case: no aspect of the contract is perceived to be enforceable, with the exception of a lump-sum transfer exchanged at the time of the agreement.⁸ The final-good producer makes a take-it-or-leave-it offer to an upstream firm chosen from a pool of potential suppliers whose number is large enough always to exceed the number of realized matches. This implies that there are always some unmatched suppliers. The offer includes a lump-sum transfer from the supplier to the buyer such that the participation constraint of the upstream firm exactly binds. It is assumed that the outside option of intermediate producers is zero, so participation in the contractual relationship leaves suppliers with zero profits. Nash bargaining over the surplus of the relationship is assumed. This leaves the upstream firm with a share ω of revenues. The upstream firm then maximizes variable profits equal to $(\omega p_s - 1)x_s$, where ω is the share of revenues going to the intermediate producer, p_s is the market price of the final good, and x_s is the amount of inputs produced. The price charged for the final good is

$$p_s = \frac{1}{\alpha\omega},$$

while sales are

$$x_s = \mathcal{AB}(\alpha\omega)^{1/(1-\alpha)}.$$

The relative magnitude of the parameters λ and ω shapes the final price and hence the size of each type of firm. If $p_v/p_s = \lambda\omega < 1$, integrated firms charge a lower price and are larger in terms of total output. If $\lambda\omega > 1$, firms that outsource produce more output because they charge a lower

⁸The setting under non-integration is also related to [Antràs \(2003\)](#), although that paper features a decision about the allocation of residual property rights, while I stick to the transaction-cost literature approach.

price. Total profits of a specialized upstream firm are

$$\Pi_m = (1 - \alpha)\omega\mathcal{AB}(\alpha\omega)^{\alpha/(1-\alpha)} - f_m - T,$$

where T is the upfront fee asked by the buyer to the supplier in order to participate in the relationship. Total profits of a downstream firm are

$$\Pi_s = (1 - \omega)\mathcal{AB}(\alpha\omega)^{\alpha/(1-\alpha)} - f_s\kappa + T,$$

and, due to the existence of a very large mass of potential suppliers, it turns out that

$$T = (1 - \alpha)\omega\mathcal{AB}(\alpha\omega)^{\alpha/(1-\alpha)} - f_m.$$

The intermediate supplier therefore earns a payoff equal to zero from the relationship, because the final-good producer secures all the surplus. Profits of the downstream firm consequently become

$$\Pi_s = \mathcal{AB}(\alpha\omega)^{\alpha/(1-\alpha)}(1 - \alpha\omega) - f_s\kappa - f_m. \quad (5)$$

2.3 Organizational efficiency

Conceptually, the upstream and downstream stages can be interpreted as distinct production divisions. Under integration, the firm operates as a unified hierarchy: this internalizes the transaction and implicitly resolves contractual frictions, but it entails governance costs ($\lambda > 1$). By contrast, outsourcing separates the two stages across firm boundaries, thereby introducing contracting costs and requiring an additional upstream labor fixed cost, f_m , alongside potentially different setup costs (f_s versus f_v).

It is straightforward to verify that neither organizational arrangement has an inherent efficiency advantage. To see this, I impose the restriction $\lambda\omega = 1$, which equalizes price and firm size across organizational forms ($p_v = p_s = p$ and $x_v = x_s = x$), and then compare total costs. Under vertical integration, total cost TC_v is

$$TC_v = f_v\kappa + \lambda x,$$

while under outsourcing, total cost TC_s is

$$TC_s = f_s\kappa + f_m + x.$$

Hence,

$$TC_v - TC_s = (\lambda - 1)x - f_m - (f_s - f_v)\kappa, \quad (6)$$

whose sign is, in general, ambiguous. This expression highlights that, keeping firms at the same size, three components determine the relative efficiency of the two organizational forms: the variable labor-cost inefficiency associated with vertical integration, $(\lambda - 1)x$; the specific fixed labor-cost burden inherent to outsourcing, f_m ; and the difference in organizational fixed costs, $(f_s - f_v)\kappa$.⁹

3 Equilibrium analysis

I now formally characterize and derive the mixed equilibrium.

Definition (Mixed Free-Entry Equilibrium). *A Mixed Free-Entry Equilibrium is a collection*

$$\{s, v, P, \mathcal{AB}\},$$

with $s > 0$ and $v > 0$, such that: (i) the price index P and the demand shifter $\mathcal{AB}$ are consistent with consumer utility maximization and with firms' demand for varieties used in fixed-capital production; and (ii) free entry and exit adjust the masses of downstream specialized firms (s) and integrated firms (v) until net profits are zero for both organizational forms, $\Pi_v(s, v) = 0$ and $\Pi_s(s, v) = 0$.

Since $\kappa = C_\xi P^{-\xi(1-\alpha)/\alpha}$, the equilibrium value of P implicitly determines the equilibrium value of κ , and vice versa. Given that total income is equal to $I = L + s\Pi_s + v\Pi_v$, the expenditure on the manufacturing sector, $\mathcal{AB} \cdot P$, is

$$\mathcal{AB} \cdot P = \xi L + \xi v [\mathcal{AB}(1 - \alpha)\rho_v - f_v\kappa] + \xi s [\mathcal{AB}(1 - \alpha\omega)\rho_s - f_s\kappa - f_m] + \xi v f_v\kappa + \xi s f_s\kappa. \quad (7)$$

The first term on the right-hand side is the expenditure share ξ applied to labor income, the next two capture the expenditure share applied to profits (the bracketed expressions are Π_v and Π_s respectively), while the last two terms capture firms' expenditure on the differentiated varieties used in the production of fixed capital. The price index, P , is

$$P(s, v) = s \rho_s + v \rho_v,$$

where $\rho_s \equiv (1/\alpha\omega)^{-\alpha/(1-\alpha)}$ and $\rho_v \equiv (\lambda/\alpha)^{-\alpha/(1-\alpha)}$ denote the contribution of one firm to the index, which depends on the organizational form.

⁹In a mixed free-entry equilibrium the zero-profit conditions will in fact equalize these costs at $\lambda\omega = 1$, through the equilibrium value of κ . The point of (6) is that the primitives alone do not rank the two arrangements.

After algebraic manipulations, the demand shifter, $\mathcal{AB}$, satisfies

$$\mathcal{AB}(s, v) = \frac{\xi(L - sf_m)}{s\rho_s[1 - \xi(1 - \alpha\omega)] + v\rho_v[1 - \xi(1 - \alpha)]} = \frac{\xi M(s)}{\mathcal{D}(s, v)}, \quad (8)$$

where

$$M(s) \equiv L - sf_m, \quad (9)$$

$$\mathcal{D}(s, v) \equiv s\rho_s[1 - \xi(1 - \alpha\omega)] + v\rho_v[1 - \xi(1 - \alpha)]. \quad (10)$$

I assume a simple entry-exit dynamics: firms enter whenever profits are positive and exit whenever profits are negative. The dynamical system is

$$\begin{cases} \dot{s} = \Pi_s(s, v), \\ \dot{v} = \Pi_v(s, v). \end{cases} \quad (11)$$

An interior steady state therefore requires $\Pi_v(s, v) = 0$ and $\Pi_s(s, v) = 0$.

From (4), the zero-profit condition under vertical integration holds if and only if

$$\Pi_v(s, v) = 0 \Leftrightarrow \mathcal{AB}(s, v) = \frac{f_v\kappa(s, v)}{(1 - \alpha)\rho_v}. \quad (12)$$

Similarly, from (5), the zero-profit condition under outsourcing requires

$$\Pi_s(s, v) = 0 \Leftrightarrow \mathcal{AB}(s, v) = \frac{f_m + f_s\kappa(s, v)}{(1 - \alpha\omega)\rho_s}. \quad (13)$$

Note that the zero-profit conditions (12) and (13) are linear relationships between the unit cost of fixed capital, $\kappa(s, v)$, and the industry demand shifter, $\mathcal{AB}(s, v)$. Figure 1 shows that their intersection determines the steady-state unit cost of fixed capital (and hence, implicitly, the price index), together with the steady-state demand shifter. The two loci are structurally asymmetric. The locus $\Pi_v = 0$ is a ray through the origin, because an integrated firm's fixed cost is entirely given by $f_v\kappa$. By contrast, the locus $\Pi_s = 0$ has a strictly positive intercept, because an outsourcing firm's fixed cost is $f_s\kappa + f_m$ and therefore includes the labor-only component f_m , which is independent of κ . From an economic point of view, this means that the fixed cost of an integrated firm is fully indexed to the unit cost of fixed capital, whereas the fixed cost of an outsourcing firm is only partially indexed to it. For the two zero-profit loci to intersect at a positive value of κ , the integrated-firm break-even demand level must rise more steeply with κ than the outsourcing-firm break-even demand level.

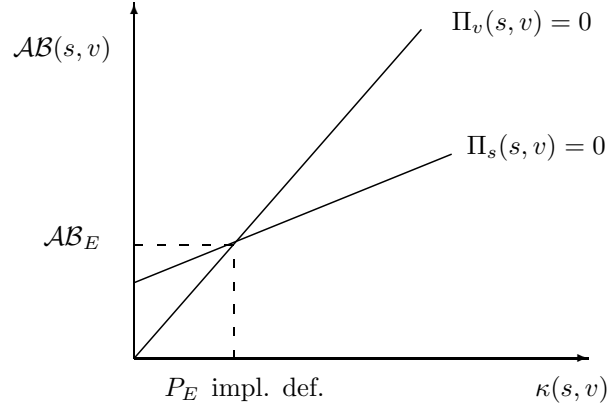


Figure 1: Graphical representation of the zero-profit conditions, $\Pi_s(s, v) = 0$ and $\Pi_v(s, v) = 0$, whose intersection gives the steady-state demand shifter, $\mathcal{AB}_E$, and implicitly defines the steady-state price index, P_E .

Solving (12) and (13) yields the equilibrium value of the unit cost of capital:

$$\kappa_E = \frac{(1 - \alpha)\rho_v f_m}{(1 - \alpha\omega)\rho_s f_v - (1 - \alpha)\rho_v f_s}.$$

Since $\kappa_E = C_\xi P_E^{-\xi(1-\alpha)/\alpha}$, the equilibrium price index is

$$P_E = \left\{ \frac{(1 - \alpha)\rho_v f_m}{C_\xi [(1 - \alpha\omega)\rho_s f_v - (1 - \alpha)\rho_v f_s]} \right\}^{-\frac{\alpha}{\xi(1-\alpha)}}. \quad (14)$$

This is well defined (and strictly positive) if and only if

$$\frac{f_s}{\rho_s} < \frac{f_v(1 - \alpha\omega)}{\rho_v(1 - \alpha)}, \quad (15)$$

which guarantees that $\Pi_v = 0$ is steeper than $\Pi_s = 0$ in the $(\kappa, \mathcal{AB})$ plane. I assume (15) throughout and interpret it further below from an economic point of view.

There are multiple combinations of s and v such that $P(s, v) = P_E$. The locus of these combinations is a straight line in the (s, v) space:

$$s \rho_s + v \rho_v = P_E. \quad (16)$$

Substituting κ_E into either zero-profit condition yields

$$\mathcal{AB}_E = \frac{f_m f_v}{(1 - \alpha\omega)\rho_s f_v - (1 - \alpha)\rho_v f_s}, \quad (17)$$

which is positive under (15).

To determine the equilibrium masses s and v , I evaluate (7) at the zero-profit steady state.

The expenditure equation becomes

$$s \{ \mathcal{AB}_E \rho_s [1 - \xi(1 - \alpha\omega)] + \xi f_m \} + v \mathcal{AB}_E \rho_v [1 - \xi(1 - \alpha)] = \xi L, \quad (18)$$

which can be written more compactly as

$$\mathcal{L}(s, v) = \xi L,$$

where $\mathcal{L}(s, v)$ denotes the left-hand side of (18).

The number of disintegrated firms, s , and vertically integrated firms, v , in the mixed free-entry equilibrium is given by the solution to the linear system (16) and (18). The solution is

$$s = \frac{\xi L - P_E \mathcal{AB}_E [1 - \xi(1 - \alpha)]}{\xi [f_m - \rho_s \mathcal{AB}_E \alpha (1 - \omega)]} \equiv s_E, \quad (19)$$

$$v = \frac{\xi f_m P_E - \rho_s \{ \xi L - P_E \mathcal{AB}_E [1 - \xi(1 - \alpha\omega)] \}}{\rho_v \xi [f_m - \rho_s \mathcal{AB}_E \alpha (1 - \omega)]} \equiv v_E. \quad (20)$$

I now discuss the comparative statics.

4 Comparative statics

In this section I present analytical results on the comparative statics of the mixed equilibrium. An increase in market size L changes the equilibrium composition in opposite directions across two parameter regimes.¹⁰ The direction of this change is governed by the relative fixed-capital burdens of outsourcing and integration, as stated formally in Proposition 1.

Proposition 1. *Define the fixed-capital burden of organizational form $z \in \{s, v\}$ as f_z/ρ_z . Conditional on the existence of a mixed free-entry equilibrium, an increase in market size L shifts the equilibrium composition toward the organizational form with the lower fixed-capital burden.*

If

$$\frac{f_s}{\rho_s} < \frac{f_v}{\rho_v},$$

then

$$\frac{\partial s_E}{\partial L} > 0, \quad \frac{\partial v_E}{\partial L} < 0.$$

If

$$\frac{f_v}{\rho_v} < \frac{f_s}{\rho_s} < \frac{1 - \alpha\omega}{1 - \alpha} \frac{f_v}{\rho_v},$$

¹⁰The two regimes are labeled Case I and Case III in the discussion that follows.

then

$$\frac{\partial s_E}{\partial L} < 0, \quad \frac{\partial v_E}{\partial L} > 0.$$

Proof. See Appendix A.2. ■

The economic content of Proposition 1 is as follows. A firm of organizational form $z \in \{s, v\}$ incurs a fixed-capital requirement f_z , but it also adds one variety to the differentiated-input sector. This variety contributes ρ_z to the aggregate price-index term $P = s\rho_s + v\rho_v$. Since $\kappa = C_\xi P^{-\xi(1-\alpha)/\alpha}$, an increase in P lowers the unit cost of fixed capital. The relevant comparison is therefore not the fixed-capital requirement f_z alone, but this requirement per unit of contribution to P , namely f_z/ρ_z . At the mixed equilibrium, the zero-profit conditions pin down both $\mathcal{AB}_E$ and P_E as functions of parameters alone, independently of L . Hence a change in L is absorbed by changing the composition of firms rather than by changing these aggregate equilibrium objects. A larger market therefore shifts the equilibrium composition toward the organizational form with the lower fixed-capital burden, while a smaller market shifts it in the opposite direction, toward the form with the higher burden. This observation has a direct implication for the existence of the mixed equilibrium. As L falls, the mass of the lower-burden form contracts and eventually reaches zero. As L rises, the mass of the higher-burden form contracts and eventually reaches zero. Each of these two corners defines one of the two bounds on L for the existence of the mixed equilibrium. Which corner corresponds to which bound therefore depends on which organizational form has the lower fixed-capital burden. This is an issue I take up in the case-by-case discussion below.

Because preferences and technologies are identical across firms and entry is free, two identical economies that integrate their goods markets without frictions replicate a single closed economy with labor endowment equal to the sum of their endowments. Proposition 1 therefore has an open-economy counterpart: frictionless goods-market integration leaves P_E , $\mathcal{AB}_E$, and κ_E unchanged and shifts the organizational composition of the integrated market toward the form with the lower fixed-capital burden. The transmission contrasts with Conconi et al. (2012), where the liberalization of product markets operates by moving prices across an organizational cutoff: here the price-type objects are fixed and the entire adjustment is compositional.

4.1 Lower fixed-capital burden of outsourcing (Case I)

The slope of (16) is smaller in absolute value than that of (18) when

$$\frac{f_s}{\rho_s} < \frac{f_v}{\rho_v}, \tag{21}$$

that is, when outsourcing has the lower fixed-capital burden. The condition on L for the existence of the mixed equilibrium is then

$$\frac{P_E \mathcal{A} \mathcal{B}_E [1 - \xi(1 - \alpha)]}{\xi} < L < \frac{P_E \mathcal{A} \mathcal{B}_E [1 - \xi(1 - \alpha\omega)]}{\xi} + \frac{P_E f_m}{\rho_s}, \quad (22)$$

illustrated in Panel (a) of Figure 2. Condition (22) follows from imposing $s > 0$ and $v > 0$ in (19) and (20): under (21), the common denominator of both expressions is positive, so the sign of each solution is determined by its numerator alone. The lower bound in (22) corresponds to the corner $s_E = 0$, while the upper bound corresponds to the corner $v_E = 0$.

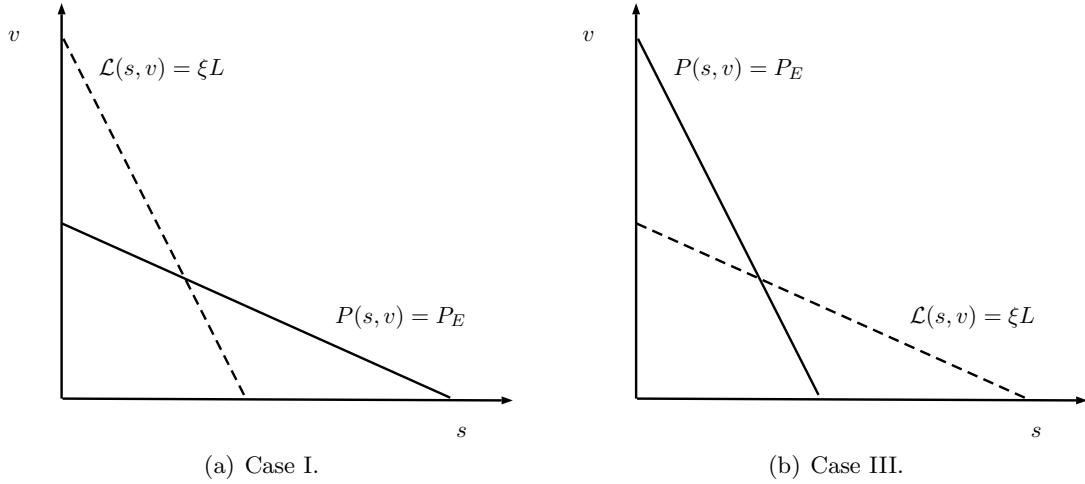


Figure 2: Interior equilibrium and relative slope.

4.2 Knife-edge condition (Case II)

In the knife-edge case

$$\frac{f_s}{\rho_s} = \frac{f_v}{\rho_v}, \quad (23)$$

the two organizational forms have the same fixed-capital burden, so (16) and (18) define two parallel lines in the (s, v) plane. A mixed equilibrium then exists only if the two lines coincide, which requires market size to take the critical value

$$L = \frac{P_E \mathcal{A} \mathcal{B}_E [1 - \xi(1 - \alpha)]}{\xi}. \quad (24)$$

At this critical value, P_E and $\mathcal{A} \mathcal{B}_E$ are still pinned down by the zero-profit conditions, but any pair (s, v) on the common locus with $s > 0$ and $v > 0$ satisfies free entry and market clearing. Hence the equilibrium composition is indeterminate, and there is a continuum of mixed equilibria. For any L that does not satisfy (24), the two loci remain parallel but distinct, and no mixed

equilibrium exists.

4.3 Lower fixed-capital burden of integration (Case III)

The slope of (16) is larger in absolute value than that of (18) when

$$\frac{f_v}{\rho_v} < \frac{f_s}{\rho_s} < \frac{1 - \alpha\omega}{1 - \alpha} \frac{f_v}{\rho_v}. \quad (25)$$

Equivalently, (25) can be written in terms of the raw fixed-cost ratio as

$$\frac{\rho_s}{\rho_v} < \frac{f_s}{f_v} < \frac{1 - \alpha\omega}{1 - \alpha} \frac{\rho_s}{\rho_v}. \quad (26)$$

Thus, integration has the lower fixed-capital burden, $f_v/\rho_v < f_s/\rho_s$, but this advantage is not so large as to violate the existence condition for a mixed equilibrium. The condition on L for the existence of the mixed equilibrium is

$$\frac{P_E \mathcal{AB}_E [1 - \xi(1 - \alpha\omega)]}{\xi} + \frac{P_E f_m}{\rho_s} < L < \frac{P_E \mathcal{AB}_E [1 - \xi(1 - \alpha)]}{\xi}, \quad (27)$$

as illustrated in Panel (b) of Figure 2. Condition (27) follows from imposing $s > 0$ and $v > 0$ in (19) and (20). Under (25), the common denominator of both expressions is negative, so positivity requires the corresponding numerators to be negative. This reverses the ordering of the two bounds relative to Case I. The lower bound in (27) corresponds to the corner $v_E = 0$, while the upper bound corresponds to the corner $s_E = 0$.

4.4 High effective fixed-capital burden of outsourcing (Case IV)

For completeness, consider

$$\frac{f_s}{\rho_s(1 - \alpha\omega)} \geq \frac{f_v}{\rho_v(1 - \alpha)}, \quad (28)$$

so that condition (15) fails.

In this case, outsourcing has a higher effective fixed-capital burden than vertical integration, where the burden is measured relative to each organizational form's operating-profit coefficient. Indeed, $\rho_s(1 - \alpha\omega)$ and $\rho_v(1 - \alpha)$ are the coefficients through which $\mathcal{AB}$ translates into operating profits under outsourcing and integration, respectively. Since outsourcing also entails the labor-only fixed cost f_m , the two zero-profit conditions do not intersect at a positive cost of capital, and hence no mixed equilibrium exists.

4.5 Discussion of comparative statics

The result in Proposition 1 differs from Grossman and Helpman (2002), where greater market thickness makes pervasive outsourcing more likely than pervasive integration only under increasing returns in matching, while under constant returns market size plays no role.¹¹ It also differs from McLaren (2000), where greater market thickness unambiguously facilitates the emergence of disintegrated firms through a mechanism based on strategic complementarity in integration decisions. Here, the key object is instead the fixed-capital burden of each organizational form. This burden depends on how much entry by a firm of a given type contributes to the price-index term P , and therefore to the reduction in the unit cost of fixed capital. As a result, larger markets need not favor outsourcing. They favor the organizational form that provides the price-index externality at the lower fixed-capital cost. In some regions of the parameter space (i.e. Case III), this form is vertical integration.

I now turn to comparative statics with respect to the production parameters. At equilibrium, the unit cost of fixed capital is

$$\kappa_E = \frac{(1 - \alpha)\rho_v f_m}{\rho_s(1 - \alpha\omega)f_v - \rho_v(1 - \alpha)f_s},$$

which is increasing in f_m and f_s , and decreasing in ω , f_v , and λ , where the dependence on λ operates through ρ_v . The mechanism is the same in each case: a change in a production parameter shifts or rotates one of the two zero-profit loci in Figure 1, and $\mathcal{AB}_E$ and κ_E adjust to restore their intersection. Consider first a reduction in f_m , which shifts the outsourcing zero-profit locus downward in parallel, since downstream non-integrated firms can now break even at a lower value of $\mathcal{AB}_E$. At the new intersection, integrated firms can also break even only if the unit cost of fixed capital falls. Hence κ_E decreases and P_E increases. Consider next an increase in ω , the share of revenues accruing to upstream non-integrated firms under Nash bargaining. A higher ω raises the operating-profit coefficient of outsourcing firms, $\rho_s(1 - \alpha\omega)$, and so the outsourcing zero-profit locus rotates clockwise and shifts downward: outsourcing firms can now break even at a lower $\mathcal{AB}_E$. As before, integrated firms then break even only if κ_E falls. The other comparative statics follow the same logic. Their effects on κ_E and $\mathcal{AB}_E$ are summarized in Table 1.

5 Local analysis of the stability of the mixed equilibrium

Given the dynamical system (11), it is possible to derive a closed-form expression for the Jacobian matrix, evaluate it at the steady-state solution, and use the signs of its eigenvalues to assess the

¹¹Even under increasing returns in matching, Grossman and Helpman (2002) finds that any mixed equilibrium with a positive mass of both outsourcing and integrated firms is unstable.

Table 1: Comparative statics of the equilibrium cost of capital and the equilibrium demand shifter with respect to production parameters.

	Equilibrium cost of capital $\kappa_E = C_\xi P_E^{-\xi(1-\alpha)/\alpha}$	Equilibrium demand shifter $\mathcal{AB}_E$
f_m	+	+
f_s	+	+
ω	−	−
f_v	−	−
λ	−	−

local stability properties of the mixed equilibrium. See Appendix A.3 for details.

5.1 Stability of the mixed equilibrium in Case I

I first characterize the stability properties of the mixed equilibrium in Case I.

Proposition 2. *In Case I, where*

$$\frac{f_s}{\rho_s} < \frac{f_v}{\rho_v},$$

the mixed equilibrium is a saddle.

Proof. Appendix A.3.2 shows that, up to a positive constant, the determinant can be factorized as follows

$$\left(\frac{f_s}{f_v} - \frac{1 - \alpha\omega}{1 - \alpha} \frac{\rho_s}{\rho_v} \right) \left(\frac{\rho_s}{\rho_v} - \frac{f_s}{f_v} \right). \quad (29)$$

In Case I then $\det(J^*) < 0$, and the two eigenvalues are real and have opposite signs, so the mixed equilibrium is a saddle. ■

When outsourcing has the lower effective fixed-capital burden than vertical integration, the mixed equilibrium is therefore a saddle, as illustrated in Figure 3.¹²

The determinant factorization given by (29) admits a clean economic reading. Under (15), integration is always the form whose break-even locus rises more steeply in the $(\kappa, \mathcal{AB})$ plane, hence the form whose profits respond more strongly to changes in the price index. This translates into the first term in (29) being always negative. Whether the mixed equilibrium is a saddle or not then turns on whether this more κ -sensitive (and more P -sensitive) form is also the cheaper one in terms of f_z/ρ_z , which is regulated by the second term in (29). In Case I the two rankings disagree (integration is more κ -sensitive but outsourcing is cheaper) and the determinant is negative; in Case III they align (integration is both more κ -sensitive and cheaper) and the determinant is

¹²The figure is illustrative and Appendix A.4 provides the argument used to derive it.

positive. Whether the resulting non-saddle in Case III is a sink depends on the trace condition analyzed below.

The misalignment of rankings in Case I has a precise economic counterpart: a deviation in the price index reverses the ordering of the break-even demand thresholds of the two organizational forms. When $P < P_E$, capital is relatively expensive, the fact that integration has a higher fixed-capital burden dominates, and there exists an intermediate region (area b in Figure 3(a)) in which outsourcing expands while integrated firms contract. When $P > P_E$, capital is relatively cheap, the fixed-capital cost advantage of outsourcing shrinks while the labor-only component f_m remains unchanged, so there is an intermediate region (area b in Figure 3(b)) in which integrated firms expand while outsourcing contracts. These opposite wedges generate the crossing vector field around the steady state that is characteristic of a saddle.

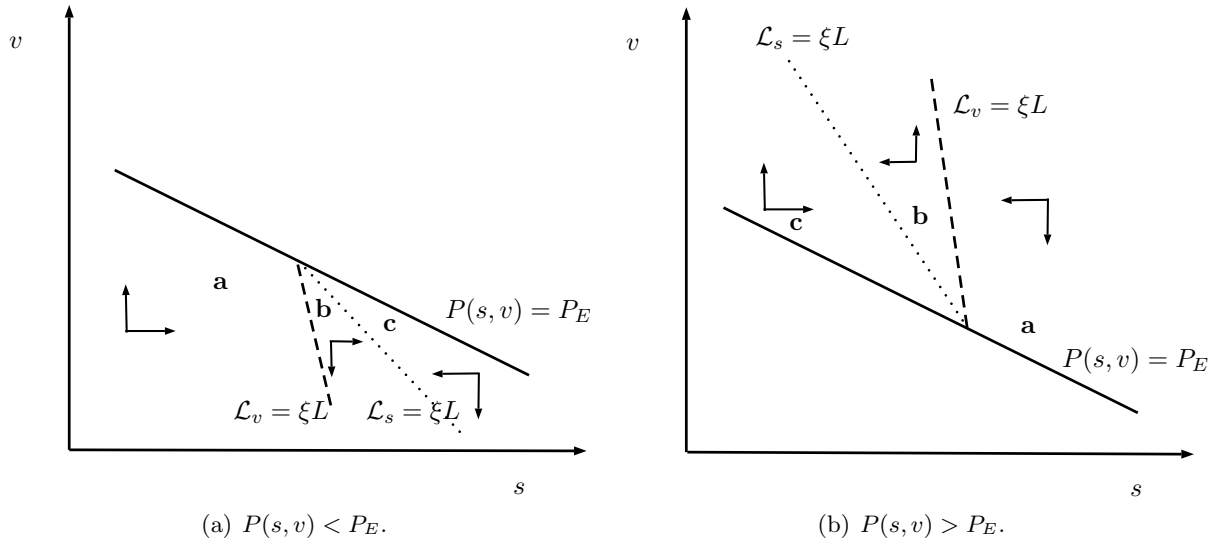


Figure 3: Adjustment incentives in Case I. Panel (a) illustrates the local sign pattern when $P(s, v) < P_E$; panel (b) illustrates the corresponding sign pattern when $P(s, v) > P_E$.

5.2 Stability of the mixed equilibrium in Case III

I now turn to characterize the local stability properties of the mixed equilibrium in Case III.

Proposition 3. *In Case III, where*

$$\frac{f_v}{\rho_v} < \frac{f_s}{\rho_s} < \frac{1 - \alpha\omega}{1 - \alpha} \frac{f_v}{\rho_v},$$

the mixed equilibrium is not a saddle. It is a sink if and only if

$$\mathcal{D}_E < \eta P_E, \tag{30}$$

where $\eta > 0$ and P_E are pinned down by primitive parameters and are independent of market size L . For $\mathcal{D}_E > \eta P_E$, the eigenvalues have positive real part and the equilibrium is locally unstable. At $\mathcal{D}_E = \eta P_E$, the linearization is non-hyperbolic.

Proof. Appendix A.3.2 shows that $\det(J^*) > 0$ in Case III, so the mixed equilibrium is not a saddle. With a positive determinant, local stability is determined by the sign of the trace. Appendix A.3.3 shows that $\eta > 0$ and that $\text{tr}(J^*) < 0$ if and only if

$$\mathcal{D}_E < \eta P_E.$$

Hence the equilibrium is a sink when this inequality holds. When the inequality is reversed, $\text{tr}(J^*) > 0$; since $\det(J^*) > 0$, the eigenvalues have positive real part and the equilibrium is locally unstable. When $\mathcal{D}_E = \eta P_E$, the trace is zero and the linearization is non-hyperbolic. ■

Figure 4 presents the phase portrait for this case.¹³

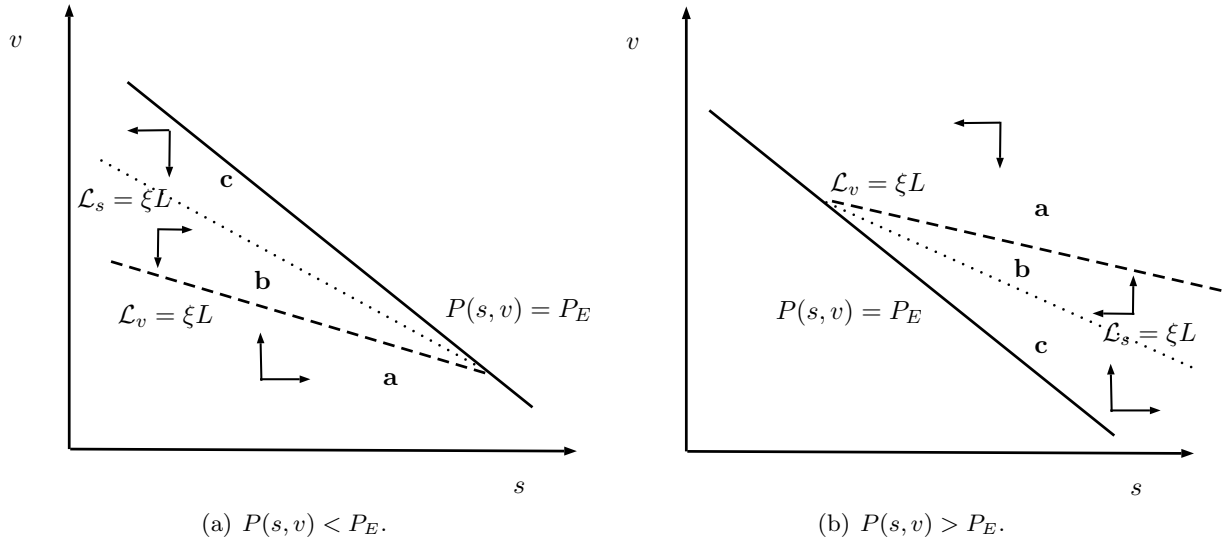


Figure 4: Adjustment incentives in Case III. Panel (a) illustrates the local sign pattern when $P(s, v) < P_E$; panel (b) illustrates the corresponding sign pattern when $P(s, v) > P_E$.

The stability condition (30) reflects the tension between two opposing forces. Entry raises the price index P , which lowers the unit cost of fixed capital κ , raises profits, and induces further entry: this is the destabilizing cost-reduction channel. At the same time, entry erodes per-firm operating margins through product-market competition: this is the stabilizing competition channel. Stability obtains when the second force dominates the first, which (30) expresses as a comparison between two objects: the threshold ηP_E , both of whose factors depend only on primitive param-

¹³The figure is illustrative and Appendix A.5 provides the characterization of the phase diagram. Contrary to Case I, the diagrammatic technique does not suffice to assess stability in Case III: here $\det(J^*) > 0$, so the regional sign analysis identifies a non-saddle pattern but cannot determine whether trajectories move toward or away from the mixed equilibrium. The trace must be computed directly.

eters; and the demand-shifter denominator $\mathcal{D}_E$, which inherits its dependence on market size L entirely through the equilibrium composition (s_E, v_E) of the number of outsourcing and vertically integrated firms. To see why, recall the recursive structure of the equilibrium. The two zero-profit conditions pin down κ_E and hence P_E as functions of primitives only, with closed form given in (14); L does not enter. The expenditure equation (18) then determines (s_E, v_E) given P_E , and it is here that L enters. The denominator $\mathcal{D}_E$, defined off-equilibrium by (10) as

$$\mathcal{D}(s, v) = s\rho_s[1 - \xi(1 - \alpha\omega)] + v\rho_v[1 - \xi(1 - \alpha)],$$

takes its equilibrium value at (s_E, v_E) and therefore moves with L , while the right-hand side of (30) stays constant.

The economic content of the condition is best seen by varying the equilibrium composition of the two organizational forms while holding the price-index term $P(s, v) = s\rho_s + v\rho_v$ at its steady-state value P_E . Substituting one outsourcing firm with ρ_s/ρ_v integrated firms preserves P_E and changes $\mathcal{D}$ by

$$\rho_s\{[1 - \xi(1 - \alpha)] - [1 - \xi(1 - \alpha\omega)]\} = \rho_s\xi\alpha(1 - \omega) > 0.$$

Two features of this expression are worth noting. First, the change vanishes at $\omega = 1$: in the absence of bargaining frictions, the two organizational forms contribute identically to $\mathcal{D}$ per unit of price-index contribution, so equilibrium composition is irrelevant for this component of the trace condition.¹⁴ The dependence of $\mathcal{D}_E$ on equilibrium composition is therefore generated by the bargaining friction $1 - \omega$. Second, the change is strictly positive: replacing an outsourcing firm by its price-index equivalent in integrated firms raises $\mathcal{D}$. Since Proposition 1 establishes that $\partial s_E/\partial L < 0$ and $\partial v_E/\partial L > 0$ in Case III, a larger market shifts the equilibrium composition away from outsourcing and toward vertical integration. Because $P_E = \rho_s s_E + \rho_v v_E$ is pinned down by the zero-profit conditions and does not vary with L , these responses satisfy $\rho_s \partial s_E/\partial L = -\rho_v \partial v_E/\partial L$: the recomposition induced by market size is precisely the substitution considered above, so the positive sign derived there applies and $\mathcal{D}_E$ rises with L . Stability requires $\mathcal{D}_E$ to remain below the threshold ηP_E ; once L pushes $\mathcal{D}_E$ above ηP_E , the destabilizing cost-reduction channel dominates and the mixed equilibrium ceases to be a sink.

Since only $\mathcal{D}_E$ depends on L , the condition $\mathcal{D}_E < \eta P_E$ can be rewritten as a threshold condition on market size.

¹⁴The limiting case $\omega = 1$ is treated in Section 7.2.

Proposition 4. *In Case III, the mixed equilibrium is a sink if and only if $L < L^*$, where*

$$L^* = \frac{P_E}{\xi} \left[\mathcal{AB}_E \eta + \frac{f_m \{ [1 - \xi(1 - \alpha)] - \eta \}}{\alpha(1 - \omega) \rho_s} \right]. \quad (31)$$

Denote by $L_{\min}$ and $L_{\max}$ the lower and upper bounds in (27). Since $\mathcal{D}_E$ is strictly increasing in L , three subcases arise:

- (a) If $\eta \geq 1 - \xi(1 - \alpha)$, then $L^* \geq L_{\max}$ and the equilibrium is a sink for every admissible L .
- (b) If $\eta \leq 1 - \xi(1 - \alpha\omega)$, then $L^* \leq L_{\min}$ and the equilibrium is locally unstable for every admissible L .
- (c) If

$$1 - \xi(1 - \alpha\omega) < \eta < 1 - \xi(1 - \alpha),$$

then $L_{\min} < L^* < L_{\max}$ and the equilibrium is a sink for $L < L^*$ and locally unstable for $L > L^*$. At $L = L^*$, the linearization is non-hyperbolic.

Proof. See Appendix A.6. ■

Figure 5 represents the three subcases as the position of $L^*(\eta)$ relative to the band $[L_{\min}, L_{\max}]$.

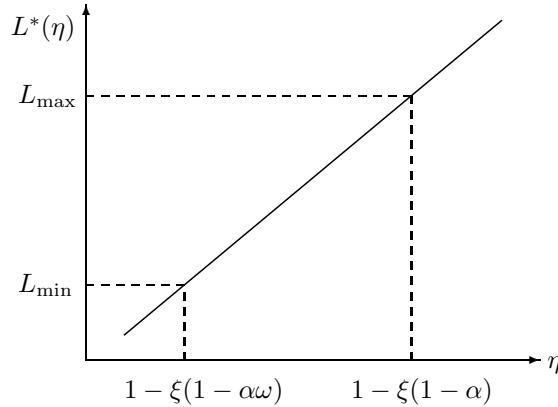


Figure 5: Graphical representation of L^* as a linear function of η , whose boundary values $\eta = 1 - \xi(1 - \alpha\omega)$ and $\eta = 1 - \xi(1 - \alpha)$ map to the endpoints $L_{\min}$ and $L_{\max}$ of the Case III existence range and demarcate the three subcases of Proposition 4.

The value of L influences the sensitivity to the cost-reduction channel, because in Case III a larger market shifts the equilibrium composition toward vertically integrated firms, whose fixed requirements depend entirely on the Cobb–Douglas composite of differentiated intermediates and are therefore fully exposed to changes in the price index. Outsourcing firms, in contrast, also incur the labor-only cost f_m , which is insensitive to P and acts as a buffer against the feedback from the cost of capital. As L rises, the share of integrated firms rises, the buffer shrinks in relative terms, and $\mathcal{D}_E$ increases. The threshold L^* identifies the market size at which $\mathcal{D}_E$ crosses the stability boundary ηP_E .

The three subcases of Proposition 4 reflect the position of L^* with respect to the admissible range of L delimited by (27). In subcase (a), η is large enough that the primitive parameters absorb the destabilizing feedback even at the largest admissible market size $L_{\max}$, so the equilibrium is a sink throughout the entire Case III existence range. In subcase (b), η is small enough that the destabilizing feedback dominates even at the smallest admissible market size $L_{\min}$, and the equilibrium is locally unstable regardless of where L sits within the existence range. In subcase (c), η lies between the two extremes and L^* falls strictly inside the admissible range: the same primitive parameters can support either a stable or an unstable mixed equilibrium depending on L , with the mechanism above centered around the share of integrated firms governing the transition.

Subcase (a), which ensures stability throughout the entire Case III existence range, admits a characterization in terms of the intensity of vertical linkages.

Proposition 5. *In Case III, the equilibrium is a sink for every admissible L whenever $\xi \leq \xi_{\text{crit}}$, where ξ_{crit} is the smaller positive root of the equation*

$$\eta(\xi) = 1 - \xi(1 - \alpha). \quad (32)$$

Proof. See Appendix A.7. ■

The parameter ξ governs the intensity of vertical linkages: it is the Cobb–Douglas share of differentiated intermediates in the production of fixed capital, as described in (2). When ξ is small, fixed capital production relies mostly on labor, so the demand for differentiated varieties generated by fixed-capital accumulation is modest. When ξ is large, the demand for varieties coming from firms’ fixed investment is substantial, and changes in the mass of active firms feed back strongly into the price index. This feedback underlies the stability mechanism. Consider the entry of a small number of firms at the mixed equilibrium. Additional firms increase variety supply, which pushes the price index P upward, thereby reducing the cost of fixed capital κ . Lower fixed costs raise profits, inducing further entry. When ξ is below ξ_{crit} , the effect is self-correcting: the increased competition on the product market erodes revenue per firm faster than the cost reduction boosts profits, so profits eventually return to zero and the system converges back to the steady state (a sink). When ξ exceeds ξ_{crit} , this sufficient bound fails: stability becomes contingent on market size, and at $L > L^*$ the cost-reduction channel dominates, entry begets more entry, and the equilibrium is locally unstable (a source), as characterized by Proposition 4.

The mixed equilibrium of Case III requires that parameters lie in the existence range (26). Provided this condition is met, I now check whether the sufficient condition for stability $\xi \leq \xi_{\text{crit}}$ binds at empirically standard substitutability values. Table 2 reports ξ_{crit} at the midpoint of the Case III existence range for the fixed-cost ratio: f_s/f_v halfway between ρ_s/ρ_v and $\frac{1-\alpha\omega}{1-\alpha} \frac{\rho_s}{\rho_v}$, for

selected configurations of $(\lambda, \omega, \alpha)$. The reported ξ_{crit} is therefore not evaluated at a common reference value of f_s/f_v across rows, since both endpoints scale with $(\lambda\omega)^{\alpha/(1-\alpha)}$. Within each value of α , four rows vary the bargaining share ω over $\{0.2, 0.6, 0.8\}$ at $\lambda = 1.2$ and add one row at $\lambda = 1.1$ with $\omega = 0.4$.

The first block of the table covers the regime where the bound is non-vacuous, $\alpha \in \{0.30, 0.40, 0.50\}$. Two patterns emerge. First, ξ_{crit} is essentially invariant to (λ, ω) : despite the midpoint of the existence range shifting substantially within each α -group, ξ_{crit} varies only at the second decimal. It is α that pins down ξ_{crit} over a large region of $(\lambda, \omega, f_s/f_v)$ space. Second, ξ_{crit} rises monotonically with α , from roughly 0.44 at $\alpha = 0.30$ to roughly 0.98 at $\alpha = 0.50$, and approaches 1 as α continues to rise. The CES elasticity $\sigma = 1/(1 - \alpha)$ ranges over the values 1.43, 1.67, and 2.00 in this block. These values are below the standard values from the gravity and firm-level trade literature. For example, [Broda and Weinstein \(2006\)](#) estimate elasticities of substitution across imported varieties and report median values between 2.2 and 3.7, depending on the period and level of aggregation. More directly at the firm level, [Bas et al. \(2017\)](#) estimate the intensive margin response under CES demand and obtain values of $-(\sigma - 1)$ around -4 , implying $\sigma \simeq 5$.

The second block therefore reports configurations with $\alpha \geq 0.55$ (equivalently, $\sigma \geq 2.22$), preserving the same (λ, ω) structure within each value of α . These values are closer to the empirical range discussed above. Throughout this block, ξ_{crit} remains strictly above one, so the bound is vacuous on the admissible domain of ξ . Hence, for the reported configurations, the mixed equilibrium is a sink at every admissible value of L , irrespective of the bargaining share, the governance-cost wedge, and the location within the Case III existence range. The transition between the binding and non-binding regimes occurs when ξ_{crit} reaches one. For the parameter configurations reported in Table 2, this threshold can be computed numerically: $\bar{\alpha}$ lies between 0.502 and 0.507, corresponding to $\bar{\sigma}$ between 2.01 and 2.03.

The Capital Flow Table evidence summarized in the Introduction and documented in Appendix A.1 establishes that the fixed-investment linkages represented by the model are quantitatively important in equipment-producing manufacturing, with sizable variation across industries. These capital-sourcing shares are not estimates of ξ : the accounts do not identify the labor employed in producing firms' fixed requirements. Nor does the stability conclusion require such an estimate: at the empirically relevant elasticities considered above, the sufficient bound of Proposition 5 is vacuous ($\xi_{\text{crit}} \geq 1$), so the mixed equilibrium is a sink for every admissible value of ξ . Stable coexistence in these parameterizations therefore requires no fine-tuning of either the elasticity of substitution or the vertical-linkage share.

Table 2: Critical value ξ_{crit} at the midpoint of the Case III range for selected parameters. In the table $\sigma \equiv 1/(1 - \alpha)$.

λ	ω	α	σ	f_s/f_v range	ξ_{crit}
1.2	0.2	0.30	1.43	[0.542, 0.728]	0.44
1.1	0.4	0.30	1.43	[0.703, 0.884]	0.44
1.2	0.6	0.30	1.43	[0.869, 1.018]	0.44
1.2	0.8	0.30	1.43	[0.983, 1.067]	0.43
1.2	0.2	0.40	1.67	[0.386, 0.592]	0.67
1.1	0.4	0.40	1.67	[0.578, 0.810]	0.67
1.2	0.6	0.40	1.67	[0.803, 1.018]	0.67
1.2	0.8	0.40	1.67	[0.973, 1.103]	0.67
1.2	0.2	0.50	2.00	[0.240, 0.432]	0.98
1.1	0.4	0.50	2.00	[0.440, 0.704]	0.98
1.2	0.6	0.50	2.00	[0.720, 1.008]	0.98
1.2	0.8	0.50	2.00	[0.960, 1.152]	0.99
1.2	0.2	0.55	2.22	[0.175, 0.346]	> 1
1.1	0.4	0.55	2.22	[0.367, 0.635]	> 1
1.2	0.6	0.55	2.22	[0.669, 0.997]	> 1
1.2	0.8	0.55	2.22	[0.951, 1.184]	> 1
1.2	0.2	0.67	3.03	[0.055, 0.145]	> 1
1.1	0.4	0.67	3.03	[0.189, 0.419]	> 1
1.2	0.6	0.67	3.03	[0.513, 0.930]	> 1
1.2	0.8	0.67	3.03	[0.920, 1.294]	> 1
1.2	0.2	0.80	5.00	[0.003, 0.014]	> 1
1.1	0.4	0.80	5.00	[0.037, 0.127]	> 1
1.2	0.6	0.80	5.00	[0.269, 0.699]	> 1
1.2	0.8	0.80	5.00	[0.849, 1.529]	> 1

6 Welfare

Since firms make zero profits in a free-entry equilibrium and consumers own firms, aggregate income at equilibrium equals labor income. With the wage normalized to one, this is L . The relevant welfare measure is therefore consumers' indirect utility. From (1),

$$V_E = L P_E^{\xi(1-\alpha)/\alpha}.$$

Using the definition of the unit cost of fixed capital, κ_E , welfare can be written equivalently as

$$V_E = \frac{C_\xi L}{\kappa_E}.$$

Conditional on the existence of a mixed equilibrium, welfare is therefore inversely related to the equilibrium unit cost of fixed capital. Since P_E and κ_E are independent of market size, an increase in L raises welfare proportionally. At the same time, market size affects the equilibrium composition

of organizational forms and, in Case III, the local stability properties of the mixed equilibrium.

The comparative statics in Table 1 translate directly into welfare effects with the opposite sign of the effects on κ_E . Within the mixed equilibrium,

$$\frac{\partial V_E}{\partial f_m} < 0, \quad \frac{\partial V_E}{\partial f_s} < 0, \quad \frac{\partial V_E}{\partial \omega} > 0, \quad \frac{\partial V_E}{\partial f_v} > 0, \quad \frac{\partial V_E}{\partial \lambda} > 0.$$

Lower fixed requirements under outsourcing, either for upstream suppliers ($f_m \downarrow$) or for downstream producers ($f_s \downarrow$), reduce the equilibrium unit cost of fixed capital and therefore raise welfare. A higher revenue share for upstream suppliers under outsourcing ($\omega \uparrow$) has the same effect. By contrast, higher fixed or variable requirements under vertical integration ($f_v \uparrow$ or $\lambda \uparrow$) also reduce κ_E and raise welfare. This last effect should not be read as a direct technological gain from making integration less efficient. It is a general-equilibrium price-index effect: conditional on coexistence, a higher relative burden of integration shifts the zero-profit intersection toward a lower cost of fixed capital, or equivalently toward a lower CES price of the differentiated composite.

These welfare effects are subject to the restrictions required for a locally stable mixed equilibrium, derived in Section 5. A welfare-improving policy must therefore preserve the existence and local stability of the mixed regime, an admissibility constraint that may bind in addition to the marginal welfare calculus.

7 Model discussion

7.1 Distinct preference and technology shares

The baseline model imposes a common Cobb–Douglas share ξ for both consumer preferences (the expenditure share on differentiated varieties) and fixed-capital technology (the share of differentiated intermediates in producing fixed capital). This assumption, adopted by [Ottaviano and Thisse \(2004\)](#) and [Ottaviano and Robert-Nicoud \(2006\)](#), delivers closed-form expressions and a transparent stability analysis. In this section I briefly discuss what happens when the two shares are allowed to differ, denoting by μ the consumer preference parameter and retaining ξ for the fixed-input technology. The distinction introduces a wedge between the demand for varieties originating from consumers and that originating from firms' fixed-capital investment. The utility function becomes $U = C_\mu D^\mu Q^{1-\mu}$, while the fixed-input production function retains $f_z = K^\xi l_f^{1-\xi}$. The consumer demand shifter is now $A = \mu I / P$, while the business demand shifter remains $B = \xi \kappa (s f_s + v f_v) / P$, and the composite demand shifter becomes

$$\mathcal{AB} = \frac{\mu I + \xi \kappa (s f_s + v f_v)}{P}.$$

The restriction $\mu = \xi$ adopted in the main analysis is mainly a tractability assumption. Relaxing it leaves the coexistence mechanism and the determinant-based case classification intact, but makes the trace condition less transparent. I summarize the main findings below (details are available upon request).

Existence. The zero-profit conditions (12) and (13) depend on the profit functions, which involve $\mathcal{AB}$ and κ but not the consumer-preference parameter μ directly. Consequently, the equilibrium cost of capital κ_E , the equilibrium price index P_E , the equilibrium demand shifter $\mathcal{AB}_E$, and the existence condition (15) are all independent of μ and coincide with the case $\mu = \xi$. The price index equation (16) is likewise unchanged. What changes is the expenditure equation (18), where the right-hand side becomes μL instead of ξL . The thresholds on f_s/f_v that define Cases I through IV are unchanged, while the bounds on the labor force L required for the existence of the mixed equilibrium are rescaled by the factor ξ/μ .

Stability: determinant. The Jacobian determinant at the mixed equilibrium retains the same sign structure as in the baseline model. In particular, it can be written as

$$\det(J^*) = \Gamma_\mu \left(\frac{f_s}{f_v} - \frac{1 - \alpha\omega}{1 - \alpha} \frac{\rho_s}{\rho_v} \right) \left(\frac{\rho_s}{\rho_v} - \frac{f_s}{f_v} \right), \quad \text{where } \Gamma_\mu > 0.$$

Although the partial derivatives of $\mathcal{AB}$ with respect to s and v acquire additional terms proportional to $(\xi - \mu)$, the combination that governs the sign of the determinant simplifies at the steady state, and the $(\xi - \mu)$ contributions cancel. As a result, Case I remains a saddle point, whereas in Case III the determinant is positive and the mixed equilibrium is not a saddle, exactly as in the baseline.

Stability: trace. The trace of the Jacobian is the one object that is quantitatively affected. When $\mu \neq \xi$, the sign of the trace still reflects the balance between the product-market competition effect and the cost-of-capital effect. But there is a structural difference: under $\mu = \xi$, the trace condition takes the form $\mathcal{D}_E/P_E < \eta$, where η depends only on primitive parameters (Proposition 3); under $\mu \neq \xi$, the analogous condition contains an additional term proportional to $(\xi - \mu)$ that depends on the equilibrium masses (s_E, v_E) themselves. The threshold separating sink from source therefore becomes composition-dependent, and Proposition 5 no longer yields the same closed-form scalar bound on ξ . Thus, Proposition 5 should be interpreted as a clean benchmark for the analytically tractable case $\mu = \xi$. Away from this restriction, the same economic force remains at work: weak vertical linkages make the cost-of-capital feedback weak relative to the product-market competition effect, whereas sufficiently strong vertical linkages may destabilize the mixed equilibrium. What is lost is the transparent closed-form market-size threshold obtained under $\mu = \xi$.

The limiting case $\xi = 0$. The separation of μ and ξ also clarifies the role of vertical linkages

by allowing an informative limiting case. Setting $\xi = 0$ (with $\mu > 0$) eliminates vertical linkages entirely: fixed capital is produced with labor only, so $\kappa = 1$ regardless of the price index. Consumer demand for varieties remains active ($A = \mu I/P > 0$), but the business component vanishes ($B = 0$). The zero-profit conditions become

$$\Pi_v = 0 \iff \mathcal{AB} = \frac{f_v}{(1-\alpha)\rho_v}, \quad \Pi_s = 0 \iff \mathcal{AB} = \frac{f_s + f_m}{(1-\alpha\omega)\rho_s},$$

where both right-hand sides are constants determined by exogenous parameters. In the $(\kappa, \mathcal{AB})$ representation, the two zero-profit conditions no longer determine a positive endogenous value of κ : they simply require $\mathcal{AB}$ to equal two constants. Generically these constants differ, so no mixed equilibrium exists; equality is a knife-edge condition. The economic reason is transparent: with $\xi = 0$, fixed costs no longer respond to the price index, so the degree of freedom that allows both organizational forms to break even simultaneously disappears. It is precisely the endogenous dependence of κ on P (operative when $\xi > 0$) that enables the coexistence of integrated and non-integrated firms. The consumer expenditure share μ , no matter how large, governs the level of demand but cannot substitute for this feedback loop through the cost of capital. The limiting case $\xi = 0$ can therefore be viewed as a frictionless counterpart to [Grossman and Helpman \(2002\)](#): it retains consumer demand, organizationally homogeneous firms, and the contractual friction associated with outsourcing, but eliminates the price-index feedback generated by vertical linkages. As in their model, each organizational form then requires a different, exogenously determined industry demand level to break even, so mixed coexistence is possible only under a knife-edge equality.¹⁵ These results show that vertical linkages are essential for robust organizational heterogeneity within the present homogeneous-firm framework.

7.2 No bargaining frictions: the limiting case $\omega = 1$

The bargaining wedge $1 - \omega$ governs the share of the variable surplus that the downstream firm retains after Nash bargaining with its upstream supplier. The limit $\omega = 1$ corresponds to a setting in which the supplier captures the entire variable surplus, so the downstream firm earns positive profits only through the lump-sum transfer T . This benchmark is informative because it isolates the role of bargaining frictions in sustaining stable organizational heterogeneity. The model remains analytically well behaved in this limit, but the qualitative picture changes in a sharp way.

Existence. At $\omega = 1$, every equilibrium object remains finite. The contribution of an outsourcing firm to the price index becomes $\rho_s = \alpha^{\alpha/(1-\alpha)}$, the equilibrium denominator $f_m - \rho_s \mathcal{AB}_E \alpha (1-\omega)$

¹⁵With frictionless matching, the two conditions coincide with the demand levels A_O and A_I in [Grossman and Helpman \(2002\)](#), equations (12) and (13), with f_v and $f_s + f_m$ in the role of their entry costs.

in (19) and (20) reduces to $f_m > 0$, and the existence condition (15) collapses to

$$\frac{f_s}{\rho_s} < \frac{f_v}{\rho_v},$$

which is the Case I slope condition (21). The Case III parameter range, defined in (25) by $f_v/\rho_v < f_s/\rho_s < [(1 - \alpha\omega)/(1 - \alpha)] \cdot f_v/\rho_v$, also collapses: the wedge $(1 - \alpha\omega)/(1 - \alpha)$ shrinks to one, and the Case III region empties out. The Case II knife-edge condition $f_s/\rho_s = f_v/\rho_v$ remains as a non-generic boundary case. Generically in parameter space, mixed equilibria therefore continue to exist at $\omega = 1$ and fall in the Case I configuration.

Stability. Substituting $\omega = 1$ into the determinant factorization (29) yields

$$\det(J^*) \propto \left(\frac{f_s}{f_v} - \frac{\rho_s}{\rho_v} \right) \left(\frac{\rho_s}{\rho_v} - \frac{f_s}{f_v} \right) = - \left(\frac{\rho_s}{\rho_v} - \frac{f_s}{f_v} \right)^2 \leq 0,$$

with equality only on the knife-edge boundary of Case II. Therefore, apart from this knife-edge case, any mixed equilibrium that exists at $\omega = 1$ is a saddle: the eigenvalues are real and of opposite sign, and stable coexistence is impossible. The market-size threshold L^* in (31) becomes irrelevant in this limit: it separates sink from source within Case III, while Case III itself collapses as $\omega \rightarrow 1$.

The intuition follows from the two-rankings argument used after Proposition 2. The condition (15) forces the integrated locus of Figure 1 to rise more steeply with κ than the outsourcing locus, so integration is always the more κ -sensitive form. Whether the determinant of the Jacobian is positive then depends on whether integration is also the cheaper form in terms of f_z/ρ_z . The bargaining-induced operating-margin wedge $(1 - \alpha)/(1 - \alpha\omega) < 1$ creates a gap between these two metrics, which is precisely what allows Case III to be non-empty: outsourcing can have a higher f_z/ρ_z than integration and still satisfy (15), because its smaller operating margin pushes its locus below integration's. At $\omega = 1$ the wedge closes, the two metrics coincide, and the parameter region in which the rankings align (i.e., Case III) disappears. Bargaining frictions are therefore not necessary for the existence of coexistence (Case I exists at $\omega = 1$), but they are necessary for the existence of a parameter region in which coexistence is locally stable.

7.3 No governance costs of integration: the limiting case $\lambda = 1$

The variable-input requirement λ captures the unit-labor-cost penalty of integrated production, which is the governance cost of internalizing the upstream stage. The limit $\lambda = 1$ corresponds to a setting in which integration is variable-cost neutral: integrated firms produce one unit of output with the same labor input as the upstream stage of an outsourcing firm. The model remains analytically well behaved in this limit, and, in contrast to the $\omega = 1$ benchmark described

above, the qualitative picture is preserved: every closed-form expression carries through with the substitution $\rho_v = \alpha^{\alpha/(1-\alpha)}$. The price ranking of integrated and outsourcing firms leads to $p_v = 1/\alpha < 1/(\alpha\omega) = p_s$, so integrated firms are the larger producers.

Existence. At $\lambda = 1$, the contribution of an integrated firm to the price index reduces to $\rho_v = \alpha^{\alpha/(1-\alpha)}$, while $\rho_s = (\alpha\omega)^{\alpha/(1-\alpha)}$ is unchanged. The ratio $\rho_s/\rho_v = \omega^{\alpha/(1-\alpha)} < 1$ for any $\omega < 1$, and the case-classification ranges of Section 3 reduce to

$$\text{Case I: } \frac{f_s}{f_v} < \omega^{\alpha/(1-\alpha)}, \quad \text{Case III: } \omega^{\alpha/(1-\alpha)} < \frac{f_s}{f_v} < \frac{1-\alpha\omega}{1-\alpha} \omega^{\alpha/(1-\alpha)}.$$

The Case III interval is non-empty whenever $\omega < 1$, so mixed equilibria continue to exist in either configuration.

Stability. The determinant factorization (29) and the trace condition of Section 5 retain their structure: λ enters only through the ratio ρ_s/ρ_v , while the bargaining wedge $(1-\alpha)/(1-\alpha\omega)$ that determines the width of the Case III window is unaffected. Case I is therefore a saddle, Case III is a non-saddle and possibly a sink, and the market-size threshold L^* in (31) continues to separate sink from source within Case III for any $\omega < 1$.

Compared with Section 7.2, the asymmetry between the two benchmarks is informative. At $\omega = 1$, Case III collapses regardless of λ , and stable coexistence becomes impossible. At $\lambda = 1$, Case III survives as long as $\omega < 1$, and stable coexistence remains possible. Bargaining frictions are therefore the structural ingredient that opens the parameter region in which the mixed equilibrium can be a sink. The variable-cost penalty of integration plays no analogous role: it is neither necessary (Case III exists at $\lambda = 1$) nor sufficient (Case III collapses at $\omega = 1$ for any λ) for stable coexistence.

7.4 Discriminating empirical predictions

The mechanism studied in this paper is complementary to explanations of coexistence based on output or productivity differences. It is nevertheless separately identifiable, because it operates on margins on which the alternatives are either silent or make different predictions. This subsection collects such predictions.

Prediction 1. *In industries where integrated and outsourcing firms coexist persistently, a larger market should raise the share of integrated firms without corresponding changes in output prices or supplier terms of trade.*

Within the mixed region, an increase in market size L leaves P_E , $\mathcal{AB}_E$, and κ_E unchanged and shifts the organizational composition toward the form with the lower fixed-capital burden f_z/ρ_z (Proposition 1). A locally stable mixed equilibrium can arise only in Case III, where integration

has the lower fixed-capital burden (Propositions 2 and 3) and where a larger market raises the share of integrated firms while leaving form-specific prices and outputs unchanged.

The constant-price market size response is not unique to the present model. In Legros and Newman (2013), when the two organizational forms coexist, the product price remains at the managerial-indifference cutoff and market size is absorbed by organizational composition. The difference lies in what sustains coexistence and determines the direction of adjustment. In their model, composition is determined by market clearing along a flat supply correspondence; here it is determined by the simultaneous zero-profit conditions of two freely entering populations, and its response to market size is governed by the relative fixed-capital burdens. The contrast is sharper with Conconi et al. (2012), away from their boundary case of coexistence, and with Alfaro et al. (2016). In the former, organizational change is triggered by changes in world prices or in supplier outside options; in the latter, tariff-driven output prices change the optimal degree of integration. Holding output prices, supplier terms of trade, and the distribution of firm productivity fixed, these mechanisms assign no independent role to market size. The tariff-based evidence of Alfaro et al. (2016) is complementary rather than competing: it identifies precisely the price channel held fixed here, and their assumption of integration costs independent of the product price is exactly the margin that vertical linkages endogenize through $\kappa(P)$. Evidence for the present mechanism would therefore be most informative if an exogenous increase in market size changed organizational composition in the direction predicted by an independently measured fixed-capital-burden ranking, without corresponding changes in the relevant price objects. Specifications that control for those objects can provide a complementary conditional test.

Empirically, market size could be represented by an exogenous shifter of potential demand or effective market access. The closest empirical counterpart of $v_E/(s_E + v_E)$ is the share of integrated producers in an initially mixed industry. Industry-level aggregates of firm-level vertical-integration indices, such as those constructed by Alfaro et al. (2016), provide a broader measure, combining changes in organizational composition with changes in the degree of integration among continuing firms. Capital-flow data of the type presented in Appendix A.1 can document the fixed-capital sourcing linkages that define the mechanism's empirical scope, but do not directly estimate ξ .¹⁶ The fixed-capital burden of each organizational form can be proxied by its annualized fixed-capital cost relative to sales.¹⁷ This mapping yields an additional cross-sectional implication: if persistent

¹⁶Section 7.1 establishes that mixed coexistence is non-generic when $\xi = 0$. It does not imply that, conditional on coexistence, the market-size response becomes smaller as ξ approaches zero from above. The capital-flow measure should therefore be interpreted as evidence that the relevant linkage exists, rather than as a variable that scales the predicted response.

¹⁷Let $F_z \equiv \kappa f_z$ denote the monetary fixed-capital cost and $R_z \equiv p_z x_z = \mathcal{AB} \rho_z$ the revenue of organizational form $z \in \{s, v\}$. Then

$$\frac{F_z}{R_z} = \frac{\kappa}{\mathcal{AB}} \frac{f_z}{\rho_z}.$$

At the mixed equilibrium, κ and $\mathcal{AB}$ are common to both forms, so their fixed-capital-cost-to-sales ranking coincides

coexistence is interpreted as evidence of local stability, integrated producers should display the lower annualized fixed-capital-cost-to-sales ratio.¹⁸

Prediction 2. *Conditional on coexistence, the firm-size ranking between integrated and outsourcing producers is governed by governance costs and the division of surplus: neither organizational form is uniformly larger.*

Formally, relative prices and outputs satisfy

$$\frac{p_v}{p_s} = \lambda\omega, \quad \frac{x_v}{x_s} = (\lambda\omega)^{-1/(1-\alpha)}.$$

Integrated firms therefore produce more output when $\lambda\omega < 1$ and less when $\lambda\omega > 1$ (Section 2.2.2). This conditional ranking does not reflect an inherent efficiency advantage of either form. At equal price and firm size ($\lambda\omega = 1$), the total-cost difference has no determinate sign before the zero-profit conditions are imposed (Section 2.3).

The alternative models, by contrast, impose rankings that do not depend on this margin. In Legros and Newman (2013), a non-integrated enterprise produces less than an integrated one whenever the two forms coexist. In Dam and Serfes (2021), integrated enterprises are precisely the high-productivity firms above the organizational cutoff, while in the sorting extension of Alfaro et al. (2016) more productive enterprises choose higher degrees of integration. Because these restrictions are predicted to hold wherever the two forms coexist, evidence that the relevant ranking weakens or becomes nonmonotonic is informative. The advantage of integrated producers in ready-mixed concrete shrinks and loses statistical significance once total sales in the local market are controlled for (Hortaçsu and Syverson, 2007). In lodging, the best- and the worst-performing properties are both disproportionately operated at arm’s length, a pattern inconsistent with a single monotone cutoff in observed performance (Lin et al., 2020). Consequently, not all the available evidence is consistent with such restrictions.

The absence of a uniform ranking in Prediction 2 is a discriminating implication rather than an unconditional restriction. A uniform ranking would not refute the present model: it could also arise if all observed markets happened to lie on the same side of $\lambda\omega = 1$. Nonuniform evidence, by contrast, challenges models that impose the same ranking wherever the two forms coexist, while remaining consistent with the present model.

with the ranking of their theoretical fixed-capital burdens f_z/ρ_z .

¹⁸Under approximately stable fundamentals, repeated coexistence without unexplained drift after generic perturbations is more consistent with a sink than with a source or saddle. Persistence is not sufficient to identify local stability, however: slow adjustment, changing fundamentals, and aggregation across distinct product markets can also generate persistent observed mixing.

8 Conclusions

This paper introduces vertical linkages arising from fixed production requirements and demonstrates the emergence of equilibria featuring organizational heterogeneity within a given industry. The analysis shows that a mixed equilibrium with both vertically integrated and outsourcing firms is stable only when two conditions are met. First, the fixed-capital burdens of the two organizational forms must be similar enough. Second, market size must be small enough and vertical linkages weak enough that the destabilizing cost-reduction channel does not dominate the stabilizing competition channel.

The mechanism generates testable implications along several margins. Within the stable mixed region, a larger market raises the share of integrated firms while leaving the equilibrium price index, the unit cost of fixed capital, and form-specific firm size unchanged. The model also imposes no uniform size ranking between the two organizational forms, while capital-flow data of the kind documented in the paper can identify industries in which the requisite fixed-capital sourcing linkages are substantial. Together, these implications distinguish the mechanism from price-theoretic and heterogeneity-based explanations of coexistence.

Within the same region, welfare rises when the fixed requirements associated with outsourcing fall or when the revenue share assigned to upstream suppliers in Nash bargaining increases. Both changes lower the equilibrium unit cost of fixed capital and the CES price of the differentiated composite. These comparative statics are not, by themselves, complete policy prescriptions: interventions large enough to move the economy outside the existence or stability region would eliminate stable mixed coexistence, while a subsidy used to offset firms' fixed costs would require financing, which the model does not account for.

To isolate the role of vertical linkages in shaping organizational heterogeneity, the analysis abstracts from several features that could interact with the mechanism. Firms are symmetric within each organizational form, ruling out productivity heterogeneity as an additional source of selection into integration. The bargaining share ω is treated as a structural parameter rather than as the endogenous outcome of a contracting game. Fixed-capital technology is restricted to the Cobb–Douglas form, so the elasticity of fixed-capital cost with respect to the price index is pinned down by ξ and α . The baseline model maintains $\mu = \xi$ to preserve the closed-form stability characterization; allowing $\mu \neq \xi$ preserves the coexistence mechanism but makes the stability condition less transparent. The analysis focuses on steady states and local dynamics around them. Organizational form is fixed at entry, so shocks change industry composition only through differential entry and exit. Allowing incumbent firms to reorganize would introduce an additional adjustment margin and, in the presence of switching costs, could generate path dependence and

hysteresis. Extending the model along these dimensions would yield a richer account of vertical structure, while the present analysis provides a tractable benchmark for understanding how vertical linkages shape organizational composition and stability.

Declaration of generative AI and AI-assisted technologies in the manuscript preparation process

During the preparation of this work, the author used OpenAI’s ChatGPT and Anthropic’s Claude to assist in checking algebraic derivations and improving the clarity and readability of the manuscript. The author independently performed and verified the derivations and reviewed and edited all AI-assisted content as needed. The author takes full responsibility for the content of the published article.

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A Appendix

A.1 Construction of the capital-flow linkage measures

This appendix documents the fixed-investment linkage measures summarized in the Introduction. I use the *180-by-123 Combined* table of the 1997 BEA Capital Flow Table ([U.S. BEA, 2003](#)), the most recent table available at this level of detail. It reports, in millions of 1997 dollars at producers' prices, purchases or leases of 180 capital commodities by 123 NAICS-based input-output industries. For each using industry, I calculate total equipment and software investment by summing lines 41–180; lines 1–40, which represent structures, are excluded.

I construct three nested sourcing measures. Let X_{ij} denote the investment of using industry j in capital commodity i , and let $\mathcal{E}$ denote the set of equipment and software commodities (lines 41–180). For each detailed industry j in the complex, let $\mathcal{C}_j^4 \subseteq \mathcal{C}_j^3 \subseteq \mathcal{C}_j^{\text{cx}} \subseteq \mathcal{E}$ denote the capital commodities whose input-output codes begin with j 's own four-digit codes, with its own three-digit code, and with the codes of the equipment complex (NAICS 333–336), respectively. The three shares reported for detailed industry j are

$$S_j^m = \frac{\sum_{i \in \mathcal{C}_j^m} X_{ij}}{\sum_{i \in \mathcal{E}} X_{ij}}, \quad m \in \{4, 3, \text{cx}\}.$$

The three-digit and equipment-complex rows sum numerators and denominators across their constituent industries. For the complex row, with $\mathcal{K}$ denoting its 20 detailed industries,

$$S^m = \frac{\sum_{j \in \mathcal{K}} \sum_{i \in \mathcal{C}_j^m} X_{ij}}{\sum_{j \in \mathcal{K}} \sum_{i \in \mathcal{E}} X_{ij}}, \quad m \in \{4, 3, \text{cx}\},$$

and the three-digit rows replace $\mathcal{K}$ with the set of their own constituents. A zero at the four-digit level indicates that the table records no purchase of a capital commodity with a matching code at the available level of detail; it does not imply the absence of all transactions between firms in that using industry.

Table [A.1](#) quantifies these linkages. For the equipment complex as a whole, purchases matched to each constituent industry's own four-digit and three-digit codes account for 3.6 and 13.8 percent of total equipment and software investment, respectively, while purchases of capital goods produced anywhere within the complex account for 61.7 percent. Across the 20 detailed industries, four-digit own-industry sourcing is concentrated in a few industries, most prominently metalworking machinery, where it reaches 30.2 percent, while the within-complex share ranges from 41.3 to 71.8 percent. At the three-digit level, own-industry sourcing is sizable in machinery (44.4 percent) and computer and electronic products (14.1 percent). By contrast, the constituent-matched four-digit share is below 10 percent in every three-digit group, ranging from 0.9 percent in electrical

equipment to 9.9 percent in machinery.

Although it comprises only four three-digit manufacturing industries, the equipment complex accounts for \$64.806 billion in equipment and software investment: 8.1 percent of the total recorded for all 123 using industries and 40.0 percent of that recorded for the 53 manufacturing industries.

These measures document the strength of fixed-capital linkages in an economically significant segment of manufacturing but do not estimate the Cobb–Douglas parameter ξ . The Capital Flow Table identifies the capital commodities acquired by each industry but not the labor employed in producing firms’ fixed requirements, and therefore cannot recover the division of fixed-capital production costs between differentiated inputs and labor that would be needed to pin down ξ .

Table A.1: Fixed-investment linkages in the equipment complex, 1997 BEA Capital Flow Table

Using industry	Total investment equip. software (\$ million)	Share of total investment (percent)		
		Own 4-digit	Own 3-digit	Within 333–336
Agriculture, construction, and mining machinery (3331)	1,588	1.0	47.8	66.9
Industrial machinery manufacturing (3332)	1,373	11.2	47.1	65.8
Commercial and service industry machinery (3333)	944	1.6	32.5	60.6
HVAC and commercial refrigeration equipment (3334)	834	0.6	45.6	67.6
Metalworking machinery manufacturing (3335)	1,707	30.2	41.4	58.7
Turbine and power transmission equipment (3336)	1,104	0.3	47.4	67.6
Other general purpose machinery (3339)	1,970	11.8	45.9	62.5
Computer and peripheral equipment (3341)	3,918	13.9	19.9	41.3
Audio, video, and communications equipment (334A)	3,687	1.8	20.5	49.2
Semiconductor and electronic components (3344)	13,833	0.0	8.9	71.8
Electronic instruments (3345)	3,736	7.0	19.6	45.7
Magnetic media manufacturing and reproducing (3346)	910	0.0	19.3	51.9
Electric lighting equipment (3351)	446	0.0	2.7	68.8
Household appliance manufacturing (3352)	554	0.0	1.8	70.0
Electrical equipment manufacturing (3353)	1,117	1.8	3.0	65.4
Other electrical equipment and components (3359)	1,703	0.8	2.1	68.9
Motor vehicle manufacturing (3361)	5,754	2.5	2.6	66.3
Motor vehicle bodies, trailers, and parts (336A)	10,713	0.1	2.4	66.6
Aerospace products and parts (3364)	7,528	4.7	5.9	51.6
Other transportation equipment (336B)	1,387	0.5	5.8	64.4
Machinery (333)	9,520	9.9	44.4	63.9
Computer and electronic products (334)	26,084	3.3	14.1	59.6
Electrical equipment (335)	3,820	0.9	2.4	68.0
Transportation equipment (336)	25,382	2.0	3.7	62.0
Equipment complex (333–336)	64,806	3.6	13.8	61.7

Notes: Total investment in equipment and software is expressed in millions of 1997 dollars at producers' prices. "Own 4-digit" matches capital commodities to the using industry's own four-digit codes, "own 3-digit" to its three-digit code, and "within 333–336" to any commodity of the equipment complex. The three-digit rows and the final row are computed from dollar totals: purchases and investment are summed across the constituent detailed industries before taking the ratio. In the own 4-digit and own 3-digit columns, each constituent's purchases are matched to its own codes. Letter-coded rows pool several 1997 NAICS industries: 334A (3342, 3343), 336A (3362, 3363), 336B (3365, 3366, 3369).

A.2 Proof of Proposition 1

From (14) and (17), both P_E and $\mathcal{AB}_E$ are independent of market size L . Differentiating (19) and (20) with respect to L then yields

$$\frac{\partial s_E}{\partial L} = \frac{1}{f_m - \rho_s \mathcal{AB}_E \alpha (1 - \omega)}, \quad \frac{\partial v_E}{\partial L} = -\frac{\rho_s}{\rho_v} \frac{\partial s_E}{\partial L}.$$

Since $\rho_s, \rho_v > 0$, the two derivatives have opposite signs, so it suffices to sign $\partial s_E / \partial L$.

Substituting the expression for $\mathcal{AB}_E$ from (17) into the denominator gives

$$\begin{aligned} f_m - \rho_s \mathcal{AB}_E \alpha (1 - \omega) &= f_m - \rho_s \alpha (1 - \omega) \frac{f_m f_v}{(1 - \alpha \omega) \rho_s f_v - (1 - \alpha) \rho_v f_s} \\ &= f_m \frac{(1 - \alpha \omega) \rho_s f_v - (1 - \alpha) \rho_v f_s - \alpha (1 - \omega) \rho_s f_v}{(1 - \alpha \omega) \rho_s f_v - (1 - \alpha) \rho_v f_s} \\ &= f_m \frac{(1 - \alpha) (\rho_s f_v - \rho_v f_s)}{(1 - \alpha \omega) \rho_s f_v - (1 - \alpha) \rho_v f_s}. \end{aligned}$$

By (15), the denominator $(1 - \alpha \omega) \rho_s f_v - (1 - \alpha) \rho_v f_s$ of the last fraction is positive. Since $f_m > 0$ and $1 - \alpha > 0$, the sign of $\partial s_E / \partial L$ is therefore the sign of $\rho_s f_v - \rho_v f_s$, equivalently of $f_v / \rho_v - f_s / \rho_s$. When $f_s / \rho_s < f_v / \rho_v$, $\partial s_E / \partial L > 0$ and $\partial v_E / \partial L < 0$. When $f_v / \rho_v < f_s / \rho_s$, $\partial s_E / \partial L < 0$ and $\partial v_E / \partial L > 0$. This proves the proposition.

A.3 Analytical characterization of the Jacobian at the mixed equilibrium

A.3.1 Computation of the Jacobian

The price index and the demand shifter can be written as

$$P(s, v) = s \rho_s + v \rho_v, \quad \mathcal{AB}(s, v) = \frac{\xi M(s)}{D(s, v)}. \quad (\text{A.1})$$

The profit functions (4) and (5) become

$$\Pi_v = \mathcal{AB}(s, v) (1 - \alpha) \rho_v - f_v \kappa(s, v), \quad (\text{A.2})$$

$$\Pi_s = \mathcal{AB}(s, v) (1 - \alpha \omega) \rho_s - f_s \kappa(s, v) - f_m. \quad (\text{A.3})$$

At the steady state, the zero-profit conditions imply

$$f_v \kappa_E = \mathcal{AB}_E (1 - \alpha) \rho_v, \quad f_m = \mathcal{AB}_E \left[(1 - \alpha \omega) \rho_s - \frac{f_s}{f_v} (1 - \alpha) \rho_v \right]. \quad (\text{A.4})$$

The relevant partial derivatives are

$$\frac{\partial P}{\partial v} = \rho_v, \quad \frac{\partial P}{\partial s} = \rho_s, \quad (\text{A.5})$$

$$\frac{\partial \mathcal{AB}}{\partial v} = -\frac{\xi M \rho_v [1 - \xi(1 - \alpha)]}{\mathcal{D}^2}, \quad (\text{A.6})$$

$$\frac{\partial \mathcal{AB}}{\partial s} = -\frac{\xi \left\{ f_m \mathcal{D} + M \rho_s [1 - \xi(1 - \alpha\omega)] \right\}}{\mathcal{D}^2}. \quad (\text{A.7})$$

From (A.2)–(A.3), each profit function depends on s and v through $\mathcal{AB}(s, v)$ and $\kappa(s, v)$. Differentiating yields

$$\frac{\partial \Pi_s}{\partial s} = (1 - \alpha\omega) \rho_s \frac{\partial \mathcal{AB}}{\partial s} + \frac{\xi(1 - \alpha)}{\alpha} \frac{f_s \kappa \rho_s}{P}, \quad (\text{A.8})$$

$$\frac{\partial \Pi_s}{\partial v} = (1 - \alpha\omega) \rho_s \frac{\partial \mathcal{AB}}{\partial v} + \frac{\xi(1 - \alpha)}{\alpha} \frac{f_s \kappa \rho_v}{P}, \quad (\text{A.9})$$

$$\frac{\partial \Pi_v}{\partial s} = (1 - \alpha) \rho_v \frac{\partial \mathcal{AB}}{\partial s} + \frac{\xi(1 - \alpha)}{\alpha} \frac{f_v \kappa \rho_s}{P}, \quad (\text{A.10})$$

$$\frac{\partial \Pi_v}{\partial v} = (1 - \alpha) \rho_v \frac{\partial \mathcal{AB}}{\partial v} + \frac{\xi(1 - \alpha)}{\alpha} \frac{f_v \kappa \rho_v}{P}. \quad (\text{A.11})$$

Each entry of the Jacobian consists of two components. The first is a negative demand-shifter effect, operating through $\partial \mathcal{AB} / \partial s < 0$ and $\partial \mathcal{AB} / \partial v < 0$: entry reduces the industry demand shifter faced by firms and therefore lowers operating profits. The second is a positive cost-of-capital effect. Entry expands the supply of varieties, raises P , and thereby lowers the unit cost of fixed capital, since $\kappa = C_\xi P^{-\xi(1-\alpha)/\alpha}$. This second channel is captured by the positive terms proportional to κ/P . The Jacobian evaluated at the steady state is

$$J^* = \begin{pmatrix} \left. \frac{\partial \Pi_s}{\partial s} \right|_E & \left. \frac{\partial \Pi_s}{\partial v} \right|_E \\ \left. \frac{\partial \Pi_v}{\partial s} \right|_E & \left. \frac{\partial \Pi_v}{\partial v} \right|_E \end{pmatrix}. \quad (\text{A.12})$$

To evaluate its entries, use the steady-state identities

$$f_v \kappa_E = \mathcal{AB}_E (1 - \alpha) \rho_v, \quad \xi M_E = \mathcal{AB}_E \mathcal{D}_E,$$

together with

$$f_m = \mathcal{AB}_E \left[(1 - \alpha\omega) \rho_s - \frac{f_s}{f_v} (1 - \alpha) \rho_v \right].$$

Then the partial derivatives of $\mathcal{AB}$ at the steady state simplify to

$$\left. \frac{\partial \mathcal{AB}}{\partial v} \right|_E = - \frac{\mathcal{AB}_E \rho_v [1 - \xi(1 - \alpha)]}{\mathcal{D}_E}, \quad (\text{A.13})$$

$$\left. \frac{\partial \mathcal{AB}}{\partial s} \right|_E = - \frac{\mathcal{AB}_E [\rho_s - \xi \frac{f_s}{f_v} (1 - \alpha) \rho_v]}{\mathcal{D}_E}. \quad (\text{A.14})$$

Substituting (A.13)–(A.14) into (A.11)–(A.8) yields

$$\left. \frac{\partial \Pi_s}{\partial s} \right|_E = \mathcal{AB}_E \rho_s \left[\frac{\xi f_s (1 - \alpha)^2 \rho_v}{\alpha f_v P_E} - \frac{(1 - \alpha \omega) \left(\rho_s - \xi \frac{f_s}{f_v} (1 - \alpha) \rho_v \right)}{\mathcal{D}_E} \right], \quad (\text{A.15})$$

$$\left. \frac{\partial \Pi_s}{\partial v} \right|_E = \mathcal{AB}_E \left[\frac{\xi f_s (1 - \alpha)^2 \rho_v^2}{\alpha f_v P_E} - \frac{(1 - \alpha \omega) \rho_s \rho_v [1 - \xi(1 - \alpha)]}{\mathcal{D}_E} \right], \quad (\text{A.16})$$

$$\left. \frac{\partial \Pi_v}{\partial s} \right|_E = \mathcal{AB}_E (1 - \alpha) \rho_v \left[\frac{\xi(1 - \alpha) \rho_s}{\alpha P_E} - \frac{\rho_s - \xi \frac{f_s}{f_v} (1 - \alpha) \rho_v}{\mathcal{D}_E} \right], \quad (\text{A.17})$$

$$\left. \frac{\partial \Pi_v}{\partial v} \right|_E = \mathcal{AB}_E (1 - \alpha) \rho_v^2 \left[\frac{\xi(1 - \alpha)}{\alpha P_E} - \frac{1 - \xi(1 - \alpha)}{\mathcal{D}_E} \right]. \quad (\text{A.18})$$

A.3.2 Computation of the determinant of the Jacobian

Using the steady-state expressions derived in Appendix A.3.1, the determinant of the Jacobian is

$$\begin{aligned} \det(J^*) &= \left. \frac{\partial \Pi_s}{\partial s} \right|_E \left. \frac{\partial \Pi_v}{\partial v} \right|_E - \left. \frac{\partial \Pi_s}{\partial v} \right|_E \left. \frac{\partial \Pi_v}{\partial s} \right|_E \\ &= \mathcal{AB}_E^2 (1 - \alpha) \rho_v^4 \left\{ \frac{\rho_s}{\rho_v} \left[\frac{\xi f_s (1 - \alpha)^2}{\alpha f_v P_E} - \frac{(1 - \alpha \omega) \left(\frac{\rho_s}{\rho_v} - \xi \frac{f_s}{f_v} (1 - \alpha) \right)}{\mathcal{D}_E} \right] \left[\frac{\xi(1 - \alpha)}{\alpha P_E} - \frac{1 - \xi(1 - \alpha)}{\mathcal{D}_E} \right] \right. \\ &\quad \left. - \left[\frac{\xi f_s (1 - \alpha)^2}{\alpha f_v P_E} - \frac{(1 - \alpha \omega) \frac{\rho_s}{\rho_v} [1 - \xi(1 - \alpha)]}{\mathcal{D}_E} \right] \left[\frac{\xi(1 - \alpha) \frac{\rho_s}{\rho_v}}{\alpha P_E} - \frac{\frac{\rho_s}{\rho_v} - \xi \frac{f_s}{f_v} (1 - \alpha)}{\mathcal{D}_E} \right] \right\}. \end{aligned} \quad (\text{A.19})$$

Expanding the two products inside braces, the terms proportional to P_E^{-2} cancel exactly, as do the terms proportional to $\mathcal{D}_E^{-2}$. The remaining terms can be regrouped as

$$\det(J^*) = \mathcal{AB}_E^2 (1 - \alpha) \rho_v^4 \left[\frac{\xi(1 - \alpha)}{\alpha P_E} \left((1 - \alpha) \frac{f_s}{f_v} - (1 - \alpha \omega) \frac{\rho_s}{\rho_v} \right) \right] \left[\frac{\xi(1 - \alpha)}{\mathcal{D}_E} \left(\frac{\rho_s}{\rho_v} - \frac{f_s}{f_v} \right) \right]. \quad (\text{A.20})$$

Rearranging terms in (A.20) yields

$$\det(J^*) = \frac{\mathcal{AB}_E^2 \xi^2 (1 - \alpha)^4 \rho_v^4}{\alpha P_E \mathcal{D}_E} \left(\frac{f_s}{f_v} - \frac{1 - \alpha \omega}{1 - \alpha} \frac{\rho_s}{\rho_v} \right) \left(\frac{\rho_s}{\rho_v} - \frac{f_s}{f_v} \right). \quad (\text{A.21})$$

The coefficient

$$\frac{\mathcal{A}\mathcal{B}_E^2 \xi^2 (1-\alpha)^4 \rho_v^4}{\alpha P_E \mathcal{D}_E}$$

is strictly positive. Therefore, the sign of $\det(J^*)$ is determined entirely by

$$\left(\frac{f_s}{f_v} - \frac{1-\alpha\omega}{1-\alpha} \frac{\rho_s}{\rho_v} \right) \left(\frac{\rho_s}{\rho_v} - \frac{f_s}{f_v} \right).$$

In Case I,

$$\frac{f_s}{f_v} < \frac{\rho_s}{\rho_v}.$$

Moreover, since

$$\frac{1-\alpha\omega}{1-\alpha} > 1,$$

we also have

$$\frac{f_s}{f_v} < \frac{1-\alpha\omega}{1-\alpha} \frac{\rho_s}{\rho_v}.$$

Hence

$$\frac{f_s}{f_v} - \frac{1-\alpha\omega}{1-\alpha} \frac{\rho_s}{\rho_v} < 0, \quad \frac{\rho_s}{\rho_v} - \frac{f_s}{f_v} > 0.$$

Therefore

$$\det(J^*) < 0.$$

Thus, in Case I, the mixed equilibrium is a saddle. This conclusion is independent of the value of $\xi > 0$: the parameter ξ enters the determinant only through a strictly positive multiplicative coefficient and therefore cannot change its sign.

In Case III,

$$\frac{\rho_s}{\rho_v} < \frac{f_s}{f_v} < \frac{1-\alpha\omega}{1-\alpha} \frac{\rho_s}{\rho_v}.$$

Hence

$$\frac{f_s}{f_v} - \frac{1-\alpha\omega}{1-\alpha} \frac{\rho_s}{\rho_v} < 0, \quad \frac{\rho_s}{\rho_v} - \frac{f_s}{f_v} < 0.$$

Therefore

$$\det(J^*) > 0.$$

Thus, in Case III, the mixed equilibrium is not a saddle: the two eigenvalues have the same sign. Local stability is then determined by the sign of the trace.

A.3.3 Computation of the trace of the Jacobian

Summing the steady-state diagonal entries of the Jacobian gives

$$\text{tr}(J^*) = \left. \frac{\partial \Pi_s}{\partial s} \right|_E + \left. \frac{\partial \Pi_v}{\partial v} \right|_E.$$

Using the expressions of these two entries obtained in Appendix A.3.1,

$$\text{tr}(J^*) = \mathcal{AB}_E \rho_v^2 \left[\frac{\xi(1-\alpha)^2 \left(\frac{\rho_s f_s}{\rho_v f_v} + 1 \right)}{\alpha P_E} - \frac{\Omega}{\mathcal{D}_E} \right],$$

where

$$\Omega \equiv (1-\alpha\omega) \left(\frac{\rho_s}{\rho_v} \right)^2 + (1-\alpha)[1-\xi(1-\alpha)] - \xi(1-\alpha)(1-\alpha\omega) \frac{\rho_s f_s}{\rho_v f_v}.$$

Since $\mathcal{AB}_E \rho_v^2 > 0$, the condition $\text{tr}(J^*) < 0$ is equivalent to

$$\frac{\xi(1-\alpha)^2 \left(\frac{\rho_s f_s}{\rho_v f_v} + 1 \right)}{\alpha P_E} < \frac{\Omega}{\mathcal{D}_E}.$$

Since all denominators are positive and, in Case III, $\Omega > 0$, define

$$\eta \equiv \frac{\alpha \Omega}{\xi(1-\alpha)^2 \left(\frac{\rho_s f_s}{\rho_v f_v} + 1 \right)} > 0.$$

Rearranging gives

$$\mathcal{D}_E < \eta P_E.$$

A.4 Phase portrait for Case I

This section provides a geometric interpretation of the saddle property of the mixed equilibrium in Case I. The formal stability result is proved in Proposition 2 through the sign of $\det(J^*)$.

Positive adjustment of firms of type $z \in \{s, v\}$ requires non-negative profits. From the zero-profit conditions,

$$\begin{cases} \Pi_s(s, v) \geq 0 & \iff \mathcal{AB}(s, v) \geq \frac{f_m + f_s \kappa(s, v)}{(1-\alpha\omega)\rho_s} \equiv \bar{\mathcal{AB}}_s(s, v), \\ \Pi_v(s, v) \geq 0 & \iff \mathcal{AB}(s, v) \geq \frac{f_v \kappa(s, v)}{(1-\alpha)\rho_v} \equiv \bar{\mathcal{AB}}_v(s, v). \end{cases} \quad (\text{A.22})$$

The thresholds $\bar{\mathcal{AB}}_s(s, v)$ and $\bar{\mathcal{AB}}_v(s, v)$ are the values of the industry demand shifter required for outsourcing and integration, respectively, to break even at the prevailing value of the price index.

Case $P(s, v) < P_E$ (*capital is expensive*). Assume that $P(s, v)$ is close to, but slightly below, P_E , so the relevant area is below the solid line in Figure 3(a). Since the unit cost of capital

$$\kappa = C_\xi P^{-\xi(1-\alpha)/\alpha}$$

is slightly above its equilibrium value, the zero-profit thresholds satisfy

$$\bar{\mathcal{A}}\mathcal{B}_s(s, v) < \bar{\mathcal{A}}\mathcal{B}_v(s, v).$$

The auxiliary line $\mathcal{L}_v(s, v) = \xi L$, which approximates $\mathcal{L}(s, v)$ in (18) when $\mathcal{A}\mathcal{B}_E$ is replaced by $\bar{\mathcal{A}}\mathcal{B}_v(s, v)$, lies below the auxiliary line $\mathcal{L}_s(s, v) = \xi L$, obtained when $\mathcal{A}\mathcal{B}_E$ is replaced by $\bar{\mathcal{A}}\mathcal{B}_s(s, v)$.

Two relevant regions can then be distinguished:

$$\bar{\mathcal{A}}\mathcal{B}_s(s, v) < \bar{\mathcal{A}}\mathcal{B}_v(s, v) \leq \mathcal{A}\mathcal{B}(s, v), \quad \text{or} \quad \bar{\mathcal{A}}\mathcal{B}_s(s, v) \leq \mathcal{A}\mathcal{B}(s, v) < \bar{\mathcal{A}}\mathcal{B}_v(s, v).$$

According to whether $\mathcal{A}\mathcal{B}(s, v)$ is above or below $\bar{\mathcal{A}}\mathcal{B}_v(s, v)$, $\dot{v}$ is positive or negative. In areas b and c of Figure 3(a), $\dot{v} < 0$ since $\mathcal{A}\mathcal{B}(s, v) < \bar{\mathcal{A}}\mathcal{B}_v(s, v)$, equivalently $\mathcal{L}_v(s, v) > \xi L$. In area a , $\dot{v} \geq 0$ since $\mathcal{A}\mathcal{B}(s, v) \geq \bar{\mathcal{A}}\mathcal{B}_v(s, v)$, equivalently $\mathcal{L}_v(s, v) \leq \xi L$.

In both areas a and b , $\mathcal{A}\mathcal{B}(s, v) > \bar{\mathcal{A}}\mathcal{B}_s(s, v)$ and $\mathcal{L}_s(s, v) < \xi L$, so $\dot{s} > 0$. When instead

$$\mathcal{A}\mathcal{B}(s, v) < \bar{\mathcal{A}}\mathcal{B}_s(s, v) < \bar{\mathcal{A}}\mathcal{B}_v(s, v),$$

we have $\mathcal{L}_s(s, v) > \xi L$: this is area c , where s is decreasing.

Case $P(s, v) > P_E$ (*capital is cheap*). Assume now that $P(s, v)$ is close to, but slightly above, P_E , so the relevant area is above the solid line in Figure 3(b). Since κ is below its equilibrium value, the ordering reverses:

$$\bar{\mathcal{A}}\mathcal{B}_v(s, v) < \bar{\mathcal{A}}\mathcal{B}_s(s, v),$$

and the auxiliary line $\mathcal{L}_v(s, v) = \xi L$ now lies above the auxiliary line $\mathcal{L}_s(s, v) = \xi L$.

Two relevant regions can then be distinguished:

$$\mathcal{A}\mathcal{B}(s, v) \leq \bar{\mathcal{A}}\mathcal{B}_v(s, v) < \bar{\mathcal{A}}\mathcal{B}_s(s, v), \quad \text{or} \quad \bar{\mathcal{A}}\mathcal{B}_v(s, v) < \mathcal{A}\mathcal{B}(s, v) \leq \bar{\mathcal{A}}\mathcal{B}_s(s, v).$$

In areas b and c of Figure 3(b), $\dot{v} > 0$ since $\mathcal{A}\mathcal{B}(s, v) \geq \bar{\mathcal{A}}\mathcal{B}_v(s, v)$, equivalently $\mathcal{L}_v(s, v) \leq \xi L$. In area a , $\dot{v} < 0$ since $\mathcal{A}\mathcal{B}(s, v) < \bar{\mathcal{A}}\mathcal{B}_v(s, v)$, equivalently $\mathcal{L}_v(s, v) > \xi L$.

In both areas a and b , $\mathcal{AB}(s, v) < \bar{\mathcal{AB}}_s(s, v)$ and $\mathcal{L}_s(s, v) > \xi L$, so $\dot{s} < 0$. When instead

$$\bar{\mathcal{AB}}_v(s, v) < \bar{\mathcal{AB}}_s(s, v) < \mathcal{AB}(s, v),$$

we have $\mathcal{L}_s(s, v) < \xi L$: this is area c , where s is increasing.

A.5 Phase portrait for Case III

This section provides a geometric interpretation of the non-saddle property of the mixed equilibrium in Case III. The formal stability result is proved in Proposition 3: in Case III, $\det(J^*) > 0$, so the mixed equilibrium is not a saddle. Whether it is a sink or a source depends on the sign of the trace, which is computed directly in the main text.

In Case III, outsourcing has a higher normalized fixed-capital burden than integration, but the mixed-equilibrium existence condition (15) still holds. As a result, the line $P(s, v) = P_E$ is steeper in absolute value than the line $\mathcal{L}(s, v) = \xi L$, reversing the relative position of the two lines compared to Case I.

The break-even thresholds $\bar{\mathcal{AB}}_s(s, v)$ and $\bar{\mathcal{AB}}_v(s, v)$, defined in (A.22), continue to govern the sign of $\dot{s}$ and $\dot{v}$.

Case $P(s, v) < P_E$ (capital is expensive). Assume that $P(s, v)$ is close to, but slightly below, P_E , so the relevant area is below the solid line in Figure 4(a). Since κ is slightly above its equilibrium value, the zero-profit thresholds satisfy

$$\bar{\mathcal{AB}}_s(s, v) < \bar{\mathcal{AB}}_v(s, v).$$

The auxiliary line $\mathcal{L}_v(s, v) = \xi L$ lies below the auxiliary line $\mathcal{L}_s(s, v) = \xi L$.

Two relevant regions can then be distinguished:

$$\bar{\mathcal{AB}}_s(s, v) < \bar{\mathcal{AB}}_v(s, v) \leq \mathcal{AB}(s, v), \quad \text{or} \quad \bar{\mathcal{AB}}_s(s, v) \leq \mathcal{AB}(s, v) < \bar{\mathcal{AB}}_v(s, v).$$

According to whether $\mathcal{AB}(s, v)$ is above or below $\bar{\mathcal{AB}}_v(s, v)$, $\dot{v}$ is positive or negative. In areas b and c of Figure 4(a), $\dot{v} < 0$ since $\mathcal{AB}(s, v) < \bar{\mathcal{AB}}_v(s, v)$, equivalently $\mathcal{L}_v(s, v) > \xi L$. In area a , $\dot{v} \geq 0$ since $\mathcal{AB}(s, v) \geq \bar{\mathcal{AB}}_v(s, v)$, equivalently $\mathcal{L}_v(s, v) \leq \xi L$.

In both areas a and b , $\mathcal{AB}(s, v) > \bar{\mathcal{AB}}_s(s, v)$ and $\mathcal{L}_s(s, v) < \xi L$, so $\dot{s} > 0$. When instead

$$\mathcal{AB}(s, v) < \bar{\mathcal{AB}}_s(s, v) < \bar{\mathcal{AB}}_v(s, v),$$

we have $\mathcal{L}_s(s, v) > \xi L$: this is area c , where s is decreasing.

Case $P(s, v) > P_E$ (*capital is cheap*). Assume now that $P(s, v)$ is close to, but slightly above, P_E , so the relevant area is above the solid line in Figure 4(b). Since κ is below its equilibrium value, the ordering reverses:

$$\bar{\mathcal{A}}\mathcal{B}_v(s, v) < \bar{\mathcal{A}}\mathcal{B}_s(s, v),$$

and the auxiliary line $\mathcal{L}_v(s, v) = \xi L$ now lies above the auxiliary line $\mathcal{L}_s(s, v) = \xi L$.

Two relevant regions can then be distinguished:

$$\mathcal{A}\mathcal{B}(s, v) \leq \bar{\mathcal{A}}\mathcal{B}_v(s, v) < \bar{\mathcal{A}}\mathcal{B}_s(s, v), \quad \text{or} \quad \bar{\mathcal{A}}\mathcal{B}_v(s, v) < \mathcal{A}\mathcal{B}(s, v) \leq \bar{\mathcal{A}}\mathcal{B}_s(s, v).$$

In areas b and c of Figure 4(b), $\dot{v} > 0$ since $\mathcal{A}\mathcal{B}(s, v) \geq \bar{\mathcal{A}}\mathcal{B}_v(s, v)$, equivalently $\mathcal{L}_v(s, v) \leq \xi L$. In area a , $\dot{v} < 0$ since $\mathcal{A}\mathcal{B}(s, v) < \bar{\mathcal{A}}\mathcal{B}_v(s, v)$, equivalently $\mathcal{L}_v(s, v) > \xi L$.

In both areas a and b , $\mathcal{A}\mathcal{B}(s, v) < \bar{\mathcal{A}}\mathcal{B}_s(s, v)$ and $\mathcal{L}_s(s, v) > \xi L$, so $\dot{s} < 0$. When instead

$$\bar{\mathcal{A}}\mathcal{B}_v(s, v) < \bar{\mathcal{A}}\mathcal{B}_s(s, v) < \mathcal{A}\mathcal{B}(s, v),$$

we have $\mathcal{L}_s(s, v) < \xi L$: this is area c , where s is increasing.

The sign pattern in Case III differs from Case I because the relative position of the lines $P(s, v) = P_E$ and $\mathcal{L}(s, v) = \xi L$ is reversed. This is the geometric counterpart of the determinant becoming positive. The mixed equilibrium is therefore not a saddle. However, this phase-portrait argument does not determine whether trajectories converge to or diverge from the mixed equilibrium. That distinction depends on the sign of $\text{tr}(J^*)$, which determines whether the real parts of the eigenvalues are negative or positive.

A.6 Proof of Proposition 4

At the mixed equilibrium,

$$\begin{aligned} \mathcal{D}_E &= s_E \rho_s [1 - \xi(1 - \alpha\omega)] + v_E \rho_v [1 - \xi(1 - \alpha)] \\ &= P_E [1 - \xi(1 - \alpha)] - \xi\alpha(1 - \omega) s_E \rho_s, \end{aligned} \tag{A.23}$$

where the second line uses the price-index identity $P_E = s_E \rho_s + v_E \rho_v$. Since P_E does not depend on L , equation (A.23) implies

$$\frac{d\mathcal{D}_E}{dL} = -\xi\alpha(1 - \omega)\rho_s \frac{\partial s_E}{\partial L}.$$

Section 3 shows that

$$\frac{\partial s_E}{\partial L} < 0 \quad \text{in Case III.}$$

Therefore $\mathcal{D}_E$ is strictly increasing in L .

By Proposition 3, in Case III the mixed equilibrium is a sink if

$$\mathcal{D}_E < \eta P_E$$

and locally unstable if

$$\mathcal{D}_E > \eta P_E.$$

The boundary between the two regimes is therefore given by

$$\mathcal{D}_E = \eta P_E.$$

Using (A.23), this condition is equivalent to

$$s_E = \frac{P_E([1 - \xi(1 - \alpha)] - \eta)}{\xi\alpha(1 - \omega)\rho_s}. \quad (\text{A.24})$$

The price-index identity gives

$$v_E \rho_v = P_E - s_E \rho_s.$$

Substituting this into the expenditure-clearing condition (18) yields

$$\xi L = P_E \mathcal{A} \mathcal{B}_E [1 - \xi(1 - \alpha)] + \xi s_E \left\{ f_m - \alpha(1 - \omega) \rho_s \mathcal{A} \mathcal{B}_E \right\}.$$

Substituting (A.24) into this expression gives

$$\begin{aligned} L^* &= \frac{P_E \mathcal{A} \mathcal{B}_E [1 - \xi(1 - \alpha)]}{\xi} + \frac{P_E([1 - \xi(1 - \alpha)] - \eta)}{\xi\alpha(1 - \omega)\rho_s} \left\{ f_m - \alpha(1 - \omega) \rho_s \mathcal{A} \mathcal{B}_E \right\} \\ &= \frac{P_E}{\xi} \left[\mathcal{A} \mathcal{B}_E \eta + \frac{f_m([1 - \xi(1 - \alpha)] - \eta)}{\alpha(1 - \omega)\rho_s} \right], \end{aligned}$$

which is (31).

Since $\mathcal{D}_E$ is strictly increasing in L , the equilibrium is a sink if and only if $L < L^*$, while it is locally unstable for $L > L^*$. At $L = L^*$, one has $\mathcal{D}_E = \eta P_E$, so $\text{tr}(J^*) = 0$. Since $\det(J^*) > 0$ in Case III, the linearization is non-hyperbolic. The proposition classifies the hyperbolic cases on either side of the threshold.

The three subcases of the proposition correspond to $L^*(\eta)$ lying above, within, or below the Case III existence range $[L_{\min}, L_{\max}]$. The boundary values of η that demarcate the subcases are therefore those at which $L^*(\eta)$ coincides with $L_{\max}$ and with $L_{\min}$. In Case III, $L_{\max}$ is the market size at which $s_E = 0$ and $L_{\min}$ is the market size at which $v_E = 0$, so each boundary η can be

obtained by evaluating the trace-zero condition $\mathcal{D}_E = \eta P_E$ at the corresponding corner of the existence range. When $s_E = 0$, the price-index identity gives $v_E \rho_v = P_E$, and (A.23) reduces to

$$\mathcal{D}_E = P_E[1 - \xi(1 - \alpha)],$$

so the boundary condition $\mathcal{D}_E = \eta P_E$ becomes $\eta = 1 - \xi(1 - \alpha)$. Substituting this value of η into (31) yields

$$L^* = \frac{P_E \mathcal{A} \mathcal{B}_E [1 - \xi(1 - \alpha)]}{\xi} = L_{\max}.$$

When $v_E = 0$, the price-index identity gives $s_E \rho_s = P_E$, and (A.23) reduces to

$$\mathcal{D}_E = P_E[1 - \xi(1 - \alpha\omega)],$$

so the boundary condition becomes $\eta = 1 - \xi(1 - \alpha\omega)$. Substituting this value of η into (31) yields

$$L^* = \frac{P_E \mathcal{A} \mathcal{B}_E [1 - \xi(1 - \alpha\omega)]}{\xi} + \frac{P_E f_m}{\rho_s} = L_{\min}.$$

Equation (31) is linear in η , with $L^* = L_{\min}$ when $\eta = 1 - \xi(1 - \alpha\omega)$ and $L^* = L_{\max}$ when $\eta = 1 - \xi(1 - \alpha)$. Since $1 - \xi(1 - \alpha\omega) < 1 - \xi(1 - \alpha)$ for $\omega < 1$, the slope of $L^*(\eta)$ is positive. Hence, if $\eta \geq 1 - \xi(1 - \alpha)$, then $L^* \geq L_{\max}$, so the equilibrium is a sink for every admissible L . If $\eta \leq 1 - \xi(1 - \alpha\omega)$, then $L^* \leq L_{\min}$, so the equilibrium is locally unstable for every admissible L . Finally, if

$$1 - \xi(1 - \alpha\omega) < \eta < 1 - \xi(1 - \alpha),$$

then $L_{\min} < L^* < L_{\max}$, and the equilibrium is a sink for $L < L^*$ and locally unstable for $L > L^*$.

A.7 Proof of Proposition 5

Let

$$u \equiv \xi(1 - \alpha), \quad r \equiv \frac{\rho_s}{\rho_v}, \quad q \equiv \frac{f_s}{f_v}, \quad K \equiv rq + 1.$$

From the definition of Ω in Appendix A.3.3,

$$\Omega = (1 - \alpha\omega)r^2 + (1 - \alpha)(1 - u) - u(1 - \alpha\omega)rq.$$

Since

$$\eta(\xi) = \frac{\alpha\Omega}{u(1 - \alpha)K},$$

the equation

$$\eta(\xi) = 1 - u$$

is equivalent, after multiplying by the positive quantity $u(1 - \alpha)K$, to

$$\alpha\Omega - u(1 - u)(1 - \alpha)K = 0.$$

Expanding Ω and collecting powers of u yields

$$\alpha\Omega - u(1 - u)(1 - \alpha)K = Au^2 + Bu + C,$$

where

$$A = (1 - \alpha)K > 0,$$

$$C = \alpha [(1 - \alpha\omega)r^2 + (1 - \alpha)] > 0,$$

and

$$B = -(1 - \alpha^2) - (1 - \alpha^2\omega)r < 0.$$

Thus the equation $\eta(\xi) = 1 - u$ is equivalent to

$$Au^2 + Bu + C = 0.$$

It is useful to record the same calculation in terms of

$$g(u) \equiv \eta(\xi) - (1 - u).$$

Indeed,

$$g(u) = \frac{\alpha\Omega}{u(1 - \alpha)K} - (1 - u) = \frac{\alpha\Omega - u(1 - u)(1 - \alpha)K}{u(1 - \alpha)K}.$$

Using the quadratic representation above, this becomes

$$g(u) = \frac{Au^2 + Bu + C}{u(1 - \alpha)K}.$$

For $u > 0$, the denominator is positive. Hence $g(u)$ has the same sign as the quadratic

$$Q(u) \equiv Au^2 + Bu + C.$$

I first show that Q has two positive real roots. Let

$$b \equiv 1 - \alpha, \quad c \equiv 1 - \alpha\omega.$$

Then

$$A = b(1 + rq), \quad C = \alpha(cr^2 + b),$$

and

$$B = -\{b(1 + \alpha) + (b + \alpha c)rq\}.$$

Hence the discriminant is

$$\Delta = \{b(1 + \alpha) + (b + \alpha c)rq\}^2 - 4\alpha b(1 + rq)(cr^2 + b).$$

In Case III, $q > r$, so $rq > r^2$. Since $c > 0$, this implies

$$cr^2 + b < crq + b.$$

Therefore

$$\begin{aligned} \Delta &> \{b(1 + \alpha) + (b + \alpha c)rq\}^2 - 4\alpha b(1 + rq)(crq + b) \\ &= [b^2 + (b - \alpha c)rq]^2 \geq 0. \end{aligned}$$

Thus $\Delta > 0$, and the quadratic has two distinct real roots. Since $A > 0$, $C > 0$, and $B < 0$, Vieta's formulas give

$$u_1 + u_2 = -\frac{B}{A} > 0, \quad u_1 u_2 = \frac{C}{A} > 0.$$

The product is positive, so the two real roots have the same sign; the sum is positive, so they cannot both be negative. Hence both roots are strictly positive. Denote them by

$$0 < u_1 < u_2.$$

I next show that the larger root lies outside the admissible range for ξ . Since $u = \xi(1 - \alpha)$, it is enough to prove that

$$u_2 > 1 - \alpha.$$

Equivalently, writing $b = 1 - \alpha$, it is enough to show that

$$u_2 > b.$$

Suppose, by contradiction, that $u_2 \leq b$. Since $A > 0$, the quadratic opens upward. Therefore, at

$u = b$, which lies weakly to the right of both roots, one must have

$$Q(b) \geq 0, \quad Q'(b) \geq 0.$$

Equivalently, since $Q(u) = A(u - u_1)(u - u_2)$,

$$Q'(b) = A\{(b - u_1) + (b - u_2)\} \geq 0.$$

A direct computation gives

$$\frac{Q(b)}{\alpha} = cr^2 + b - 2b^2 - b(b + c)rq.$$

Define

$$H \equiv crq + b - 2b^2 - b(b + c)rq.$$

Since $rq > r^2$ and $c > 0$, we have

$$\frac{Q(b)}{\alpha} < H.$$

Therefore $Q(b) \geq 0$ implies $H > 0$. On the other hand,

$$Q'(b) = b(3b - 2) + rq\{2b^2 + b(c - 1) - c\},$$

and direct expansion gives

$$H + Q'(b) = -b(1 - b)(1 + rq) < 0.$$

Hence $H > 0$ implies $Q'(b) < 0$, contradicting $Q'(b) \geq 0$. Therefore

$$u_2 > b = 1 - \alpha.$$

It follows that

$$\frac{u_2}{1 - \alpha} > 1.$$

The larger root is therefore associated with a value of ξ strictly greater than one, and hence lies outside the admissible range.

I can now sign $g(u)$. Since $A > 0$, the quadratic $Q(u)$ is positive outside the interval between its two roots and negative inside it. Therefore

$$Q(u) \geq 0 \quad \text{for all } u \in (0, u_1],$$

and, because $g(u)$ has the same sign as $Q(u)$ for $u > 0$,

$$g(u) \geq 0 \quad \text{for all } u \in (0, u_1].$$

Equivalently,

$$\eta(\xi) \geq 1 - \xi(1 - \alpha) \quad \text{for all admissible } \xi \leq \frac{u_1}{1 - \alpha}.$$

Define

$$\xi_{\text{crit}} \equiv \frac{u_1}{1 - \alpha}.$$

Then, for every admissible $\xi \leq \xi_{\text{crit}}$,

$$\eta(\xi) \geq 1 - \xi(1 - \alpha).$$

By Proposition 4, subcase (a), this implies that the mixed equilibrium is a sink for every admissible L .

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